SIAM Conference on Computational Science and Engineering (CSE07)

A Posteriori Error Estimation and Adaptivity in Computational Science and Engineering

A Three-Dimensional Self-Adaptive Goal-Oriented hp-Finite Element Method with a Multigrid Solver: Applications to Electromagnetics.

D. Pardo, L. Demkowicz, C. Torres-Verdín, M. Paszynski

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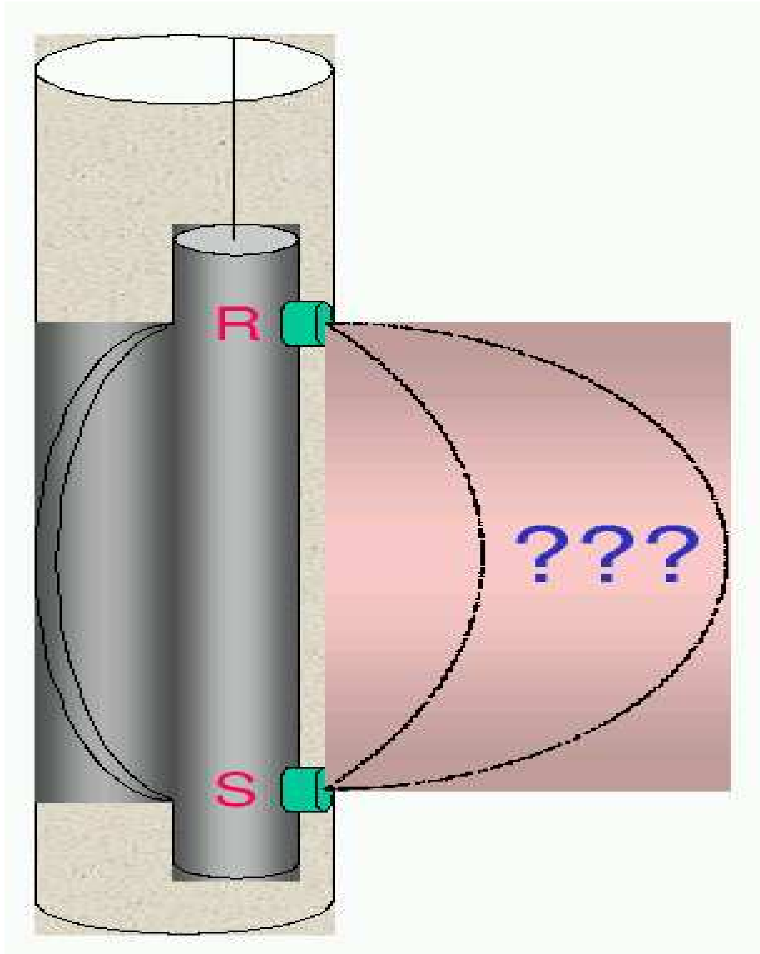
Costa Mesa, California

Department of Petroleum and Geosystems Engineering, and
Institute for Computational Engineering and Sciences (ICES)

THE UNIVERSITY OF TEXAS AT AUSTIN

MOTIVATION (APPLICATIONS)

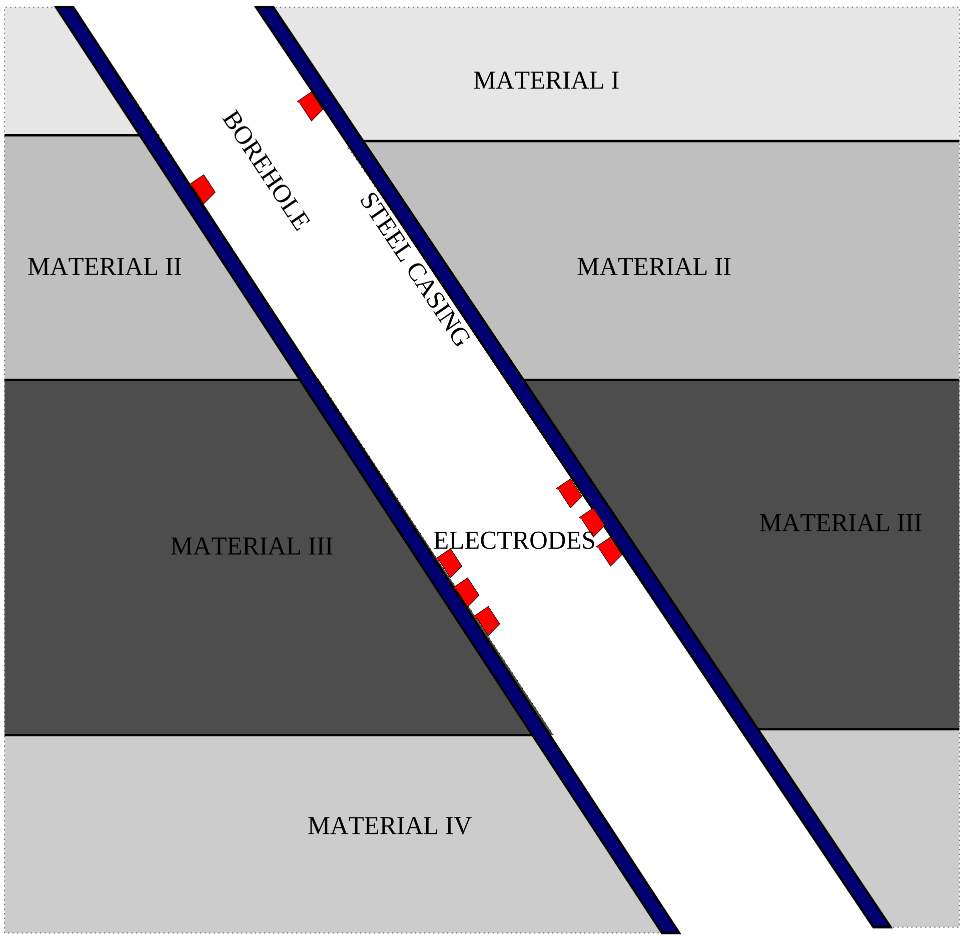
Main Objective: To Solve an Inverse Problem



A software for solving the DIRECT problem is essential in order to solve the INVERSE problem

MOTIVATION (APPLICATIONS)

Deviated Cased Wells: $\nabla\sigma\nabla u = f$ (Electrostatics)



Large material contrasts.

Large dynamic range.

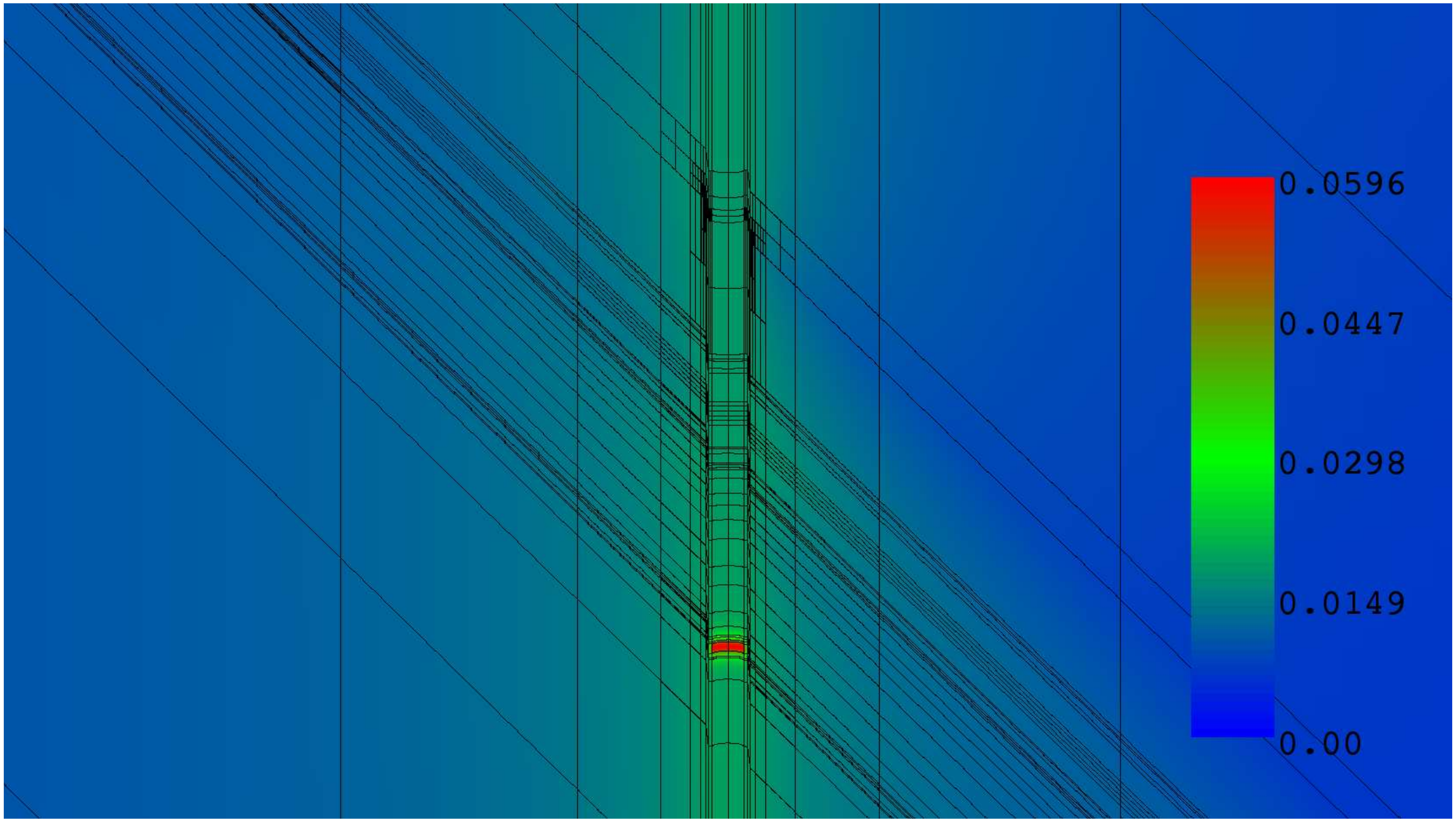
Large domain needed.

Elongated geometries.

Objective: Determine 2nd difference of potential at the receiver electrodes.

MOTIVATION (APPLICATIONS)

60 degrees deviated well



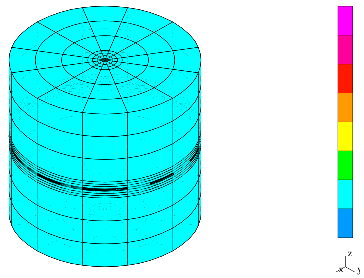
Elongated elements needed (due to geometry)
Goal-oriented adaptivity needed

OVERVIEW

1. Motivation: Simulation of resistivity logging measurements.
2. The hp-Finite Element Method.
3. Goal-Oriented hp-Adaptivity.
4. Two-Grid Solver.
   - Formulation.
   - Convergence theory.
   - The idea of goal-oriented solver.
   - The problem of elongated elements.
   - Implementation details.
5. Numerical Results.
6. Conclusions and future work.

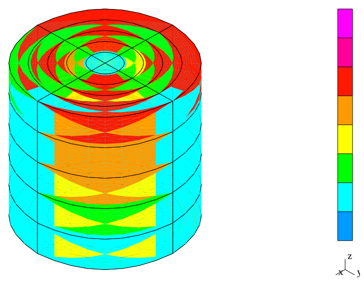
THE hp-FINITE ELEMENT METHOD (FEM)

The h-Finite Element Method



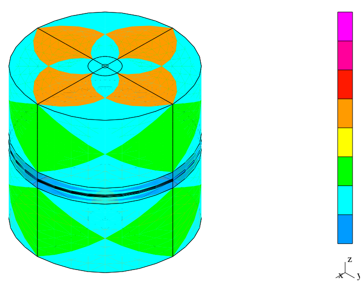
1. Convergence limited by the polynomial degree, and large material contrasts.
2. Optimal h-grids do NOT converge exponentially in real applications.
3. They may "lock" (100$\%$ error).

The p-Finite Element Method



1. Exponential convergence feasible for analytical ("nice") solutions.
2. Optimal p-grids do NOT converge exponentially in real applications.
3. If initial h-grid is not adequate, the p-method will fail miserably.

The hp-Finite Element Method



1. Exponential convergence feasible for ALL solutions.
2. Optimal hp-grids DO converge exponentially in real applications.
3. If initial hp-grid is not adequate, results will still be great.

GOAL-ORIENTED hp-ADAPTIVITY

Mathematical Formulation (Goal-Oriented Adaptivity)

Let's L be the quantity of interest (Ex.: Second difference of potential).

We consider the following problem (in variational form):

$$\left\{ \begin{array}{l} \text{Find } L(u)\text{, where } u \in V = H^1(\Omega) \text{ such that :} \\ \underbrace{\int_\Omega \sigma \nabla u \cdot \nabla v}_{b(u,v)} = \underbrace{\int_\Omega f \cdot v}_{F(v)} \quad \forall v \in V \ . \end{array} \right.$$

We define residual $r_e(v) := b(e,v)$. We seek for a functional $G \in V'' \sim V$ relating the residual and the error in the quantity of interest, that is, such that $G(r_e) := L(e)$. Functional G is the solution of the so-called *dual* problem:

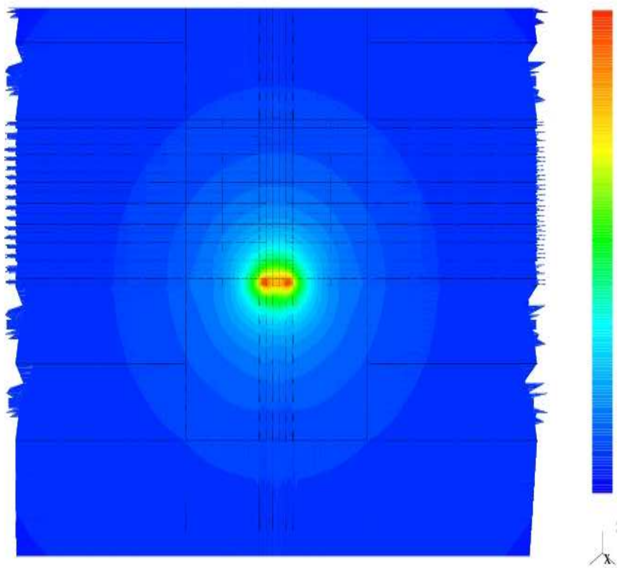
$$\left\{ \begin{array}{l} \text{Find } G \in V \text{ such that :} \\ b(v,G) = L(v) \quad \forall v \in V \ . \end{array} \right.$$

Notice that, in particular, $L(e) = b(e,G) \quad \longleftarrow \quad$ (Representation Formula).

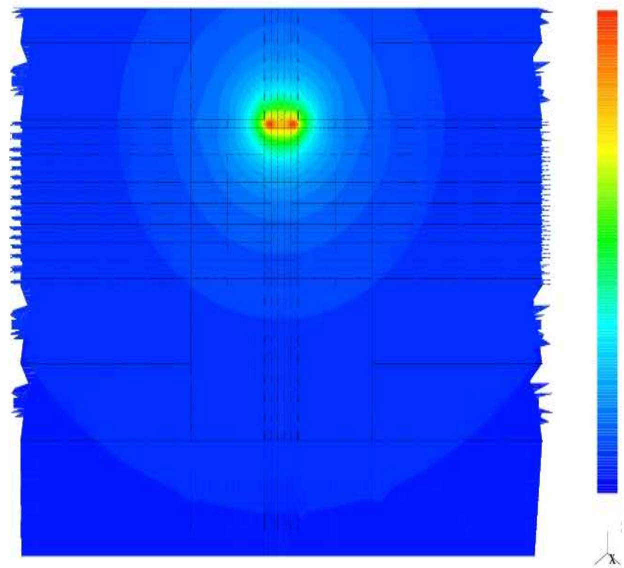
GOAL-ORIENTED hp-ADAPTIVITY

Mathematical Formulation (Goal-Oriented Adaptivity)

DIRECT PROBLEM - u -
2D Cross-Section



DUAL PROBLEM - G -
2D Cross-Section



Representation Formula for the Error in the Quantity of Interest:

$$L(u) = b(u, G) = \int_{\Omega} \sigma \nabla u \cdot \nabla G dV$$

GOAL-ORIENTED hp-ADAPTIVITY

Mathematical Formulation (Goal-Oriented Adaptivity)

We define: $e = u - u_{hp}$, $\quad u$ **exact solution of direct problem**

$\epsilon = G - G_{hp}$. $\quad G$ **exact solution of dual problem**

Upper Bound for the Error in the Quantity of Interest:

$$|L(e)| = |b(e,G)| = |b(e,\epsilon)| = |\int_\Omega \sigma\, \nabla e \cdot \nabla \epsilon \, dV| \leq$$

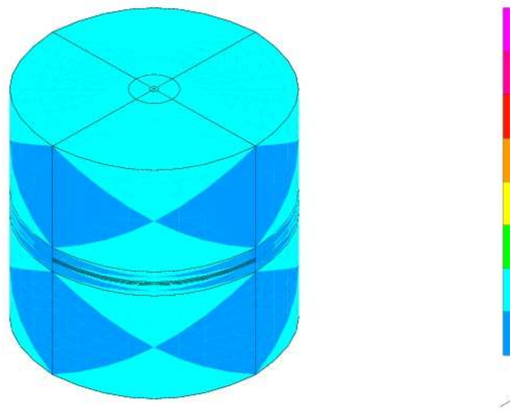
ALGORITHM I: $\displaystyle\sum_K |\int_K \sigma\, \nabla e \cdot \nabla \epsilon \, dV| \leq$

ALGORITHM II: $\displaystyle\sum_K \sqrt{\int_K \sigma\, (\nabla e)^2 \, dV}\sqrt{\int_K \sigma\, (\nabla \epsilon)^2 \, dV}$

SELF-ADAPTIVE GOAL-ORIENTED hp-FEM

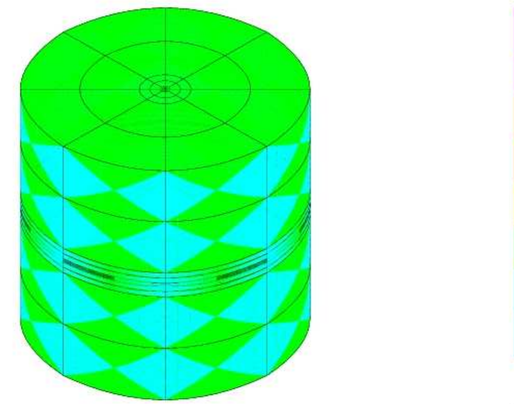
Algorithm for Goal-Oriented Adaptivity - STEP I -

Solve Direct and Dual Problems on Grid hp



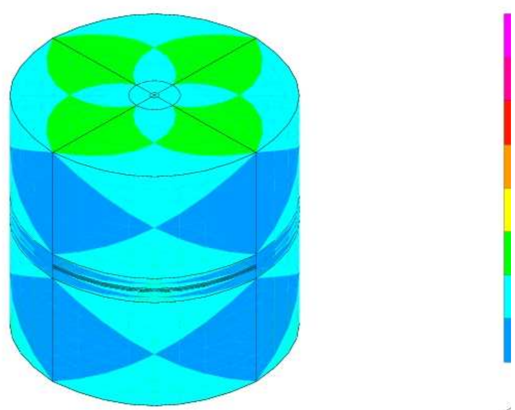
$\longrightarrow$

Solve Direct and Dual Problems on Grid $h/2, p+1$



Use the fine grid solution to estimate the coarse grid error function.
Apply the fully automatic goal-oriented hp-adaptive algorithm.

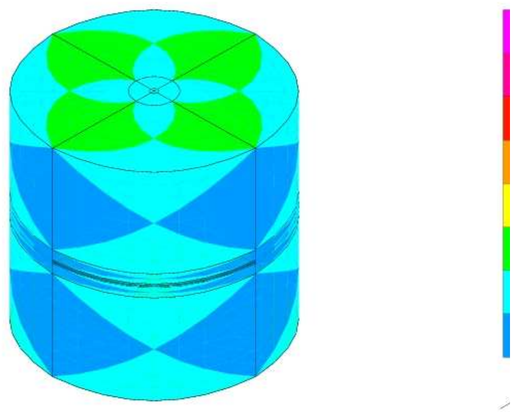
Next optimal hp-grid:



SELF-ADAPTIVE GOAL-ORIENTED hp-FEM

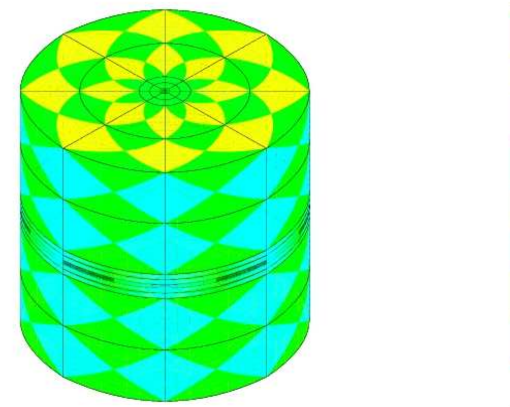
Algorithm for Goal-Oriented Adaptivity - STEP II -

Solve Direct and Dual Problems on Grid hp



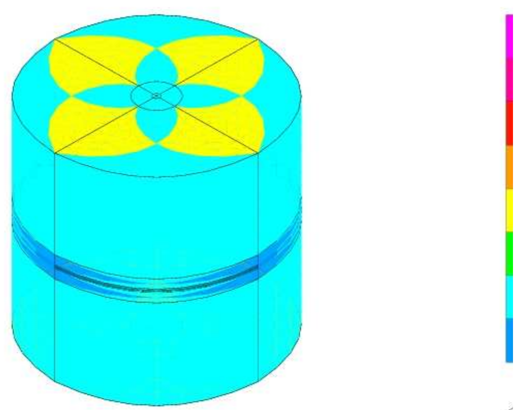
$\longrightarrow$

Solve Direct and Dual Problems on Grid $h/2, p+1$



Use the fine grid solution to estimate the coarse grid error function.
Apply the fully automatic goal-oriented hp-adaptive algorithm.

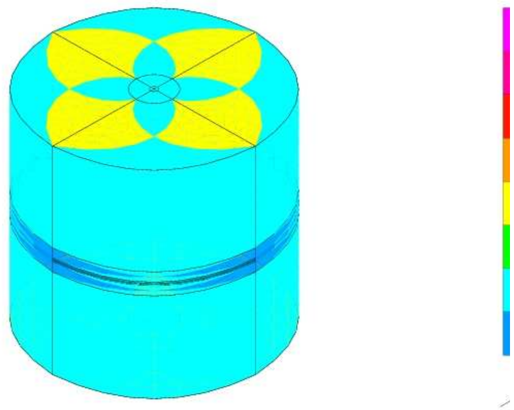
Next optimal hp-grid:



SELF-ADAPTIVE GOAL-ORIENTED hp-FEM

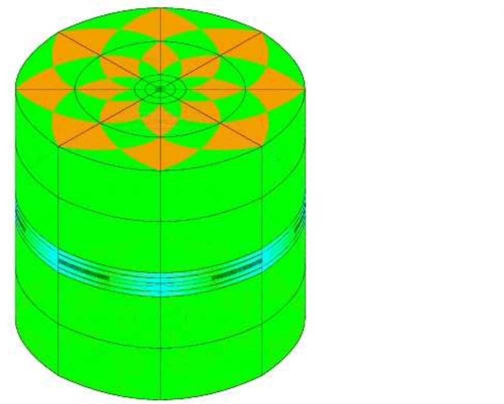
Algorithm for Goal-Oriented Adaptivity - STEP III -

Solve Direct and Dual Problems on Grid hp



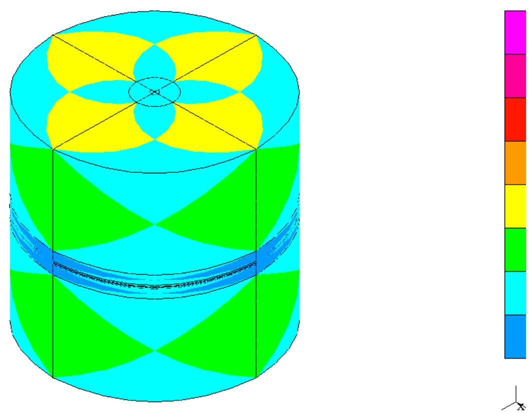
$\longrightarrow$

Solve Direct and Dual Problems on Grid $h/2, p+1$



Use the fine grid solution to estimate the coarse grid error function.
Apply the fully automatic goal-oriented hp-adaptive algorithm.

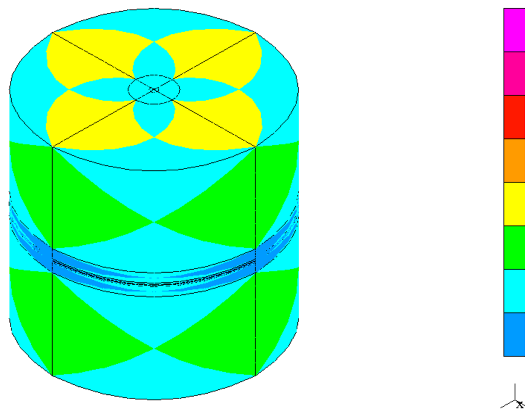
Next optimal hp-grid:



SELF-ADAPTIVE GOAL-ORIENTED hp-FEM

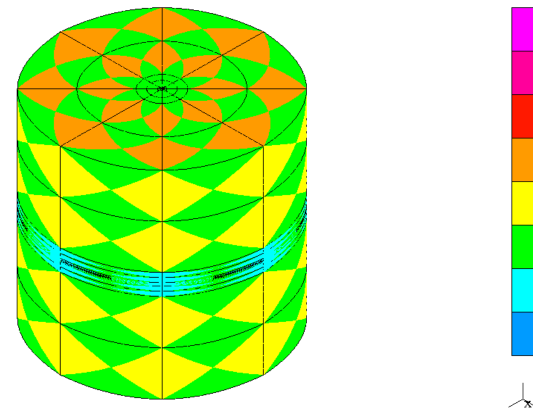
Algorithm for Goal-Oriented Adaptivity - STEP IV -

Solve Direct and Dual Problems on Grid hp



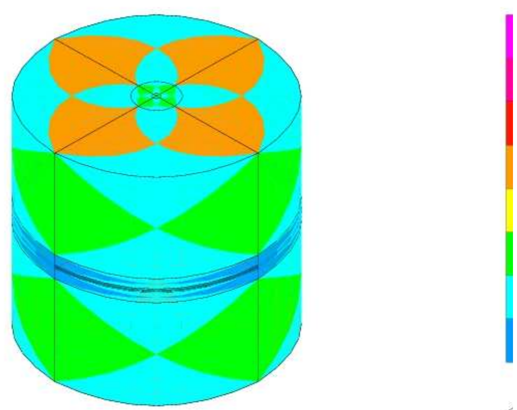
$\longrightarrow$

Solve Direct and Dual Problems on Grid $h/2, p+1$



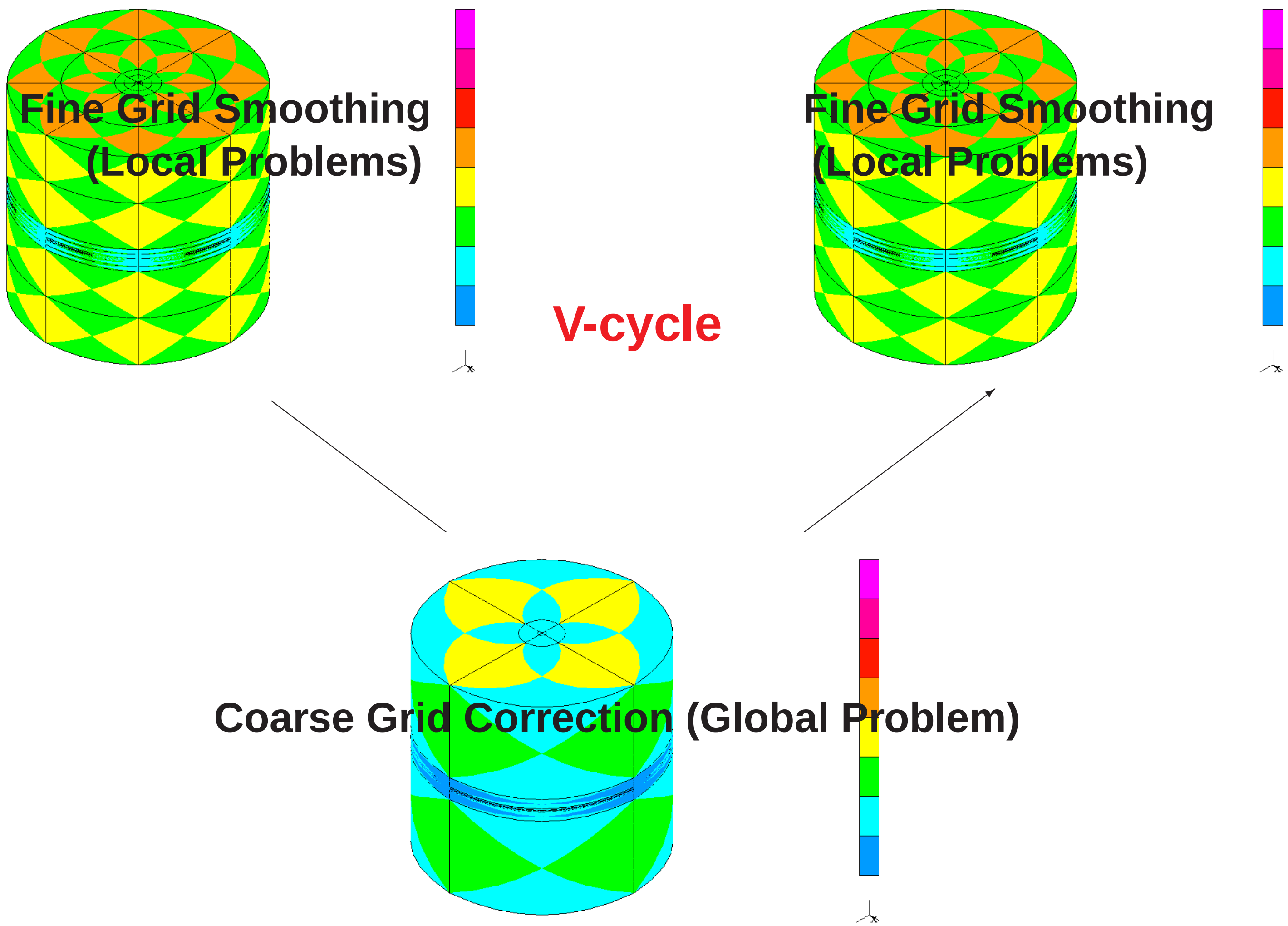
Use the fine grid solution to estimate the coarse grid error function.
Apply the fully automatic goal-oriented hp-adaptive algorithm.

Next optimal hp-grid:



TWO-GRID (TG) SOLVER: FORMULATION

Two-Grid Solver (Ax=b)



TWO-GRID (TG) SOLVER: FORMULATION

We seek x such that $Ax = b$. Consider the following iterative scheme:

$$\boldsymbol{r}^{(n+1)} = [\boldsymbol{I} - \boldsymbol{\alpha}^{(n)}\boldsymbol{A}\boldsymbol{S}]\boldsymbol{r}^{(n)}$$
$$\boldsymbol{x}^{(n+1)} = \boldsymbol{x}^{(n)} + \boldsymbol{\alpha}^{(n)}\boldsymbol{S}\boldsymbol{r}^{(n)}$$

where S is a matrix, and $\alpha^{(n)}$ is a relaxation parameter. $\alpha^{(n)}$ *optimal* if:

$$\boldsymbol{\alpha}^{(n)} = \arg\ \min \| \boldsymbol{x}^{(n+1)} - \boldsymbol{x} \|_{\boldsymbol{A}} = \frac{(\boldsymbol{A}^{-1}\boldsymbol{r}^{(n)}, \boldsymbol{S}\boldsymbol{r}^{(n)})_{\boldsymbol{A}}}{(\boldsymbol{S}\boldsymbol{r}^{(n)}, \boldsymbol{S}\boldsymbol{r}^{(n)})_{\boldsymbol{A}}}$$

Then, we define our two grid solver as:

1 iteration with $S = S_F = \sum A_i^{-1}$ $\quad +$

1 iteration with $S = S_C = P_C A_C^{-1} R_C$

TG SOLVER: CONVERGENCE THEORY

Error reduction and stopping criteria

Error step n **=** $e^{(n)} = x^{(n)} - x$**.** $\tilde{e}^{(n)} = [I - S_C A]e^{(n)} = [I - P_C]e^{(n)}$**. Then:**

$$\frac{\| e^{(n+1)} \|_A^2}{\| e^{(n)} \|_A^2} = 1 - \frac{| (\tilde{e}^{(n)}, S_F A \tilde{e}^{(n)})_A |^2}{\| \tilde{e}^{(n)} \|_A^2 \| S_F A \tilde{e}^{(n)} \|_A^2} = 1 - \frac{| (\tilde{e}^{(n)}, (P_C + S_F A)\tilde{e}^{(n)})_A |^2}{\| \tilde{e}^{(n)} \|_A^2 \| S_F A \tilde{e}^{(n)} \|_A^2}$$

Then:

$$\frac{\| e^{(n+1)} \|_A^2}{\| e^{(n)} \|_A^2} \leq \sup_e \left[1 - \frac{| (e, (P_C + S_F A)e)_A |^2}{\| e \|_A^2 \| S_F A e \|_A^2}\right] \leq C < 1$$

(Error Reduction)

For our stopping criteria, we want: Solver Error $\approx$ Discretization Error. That is:

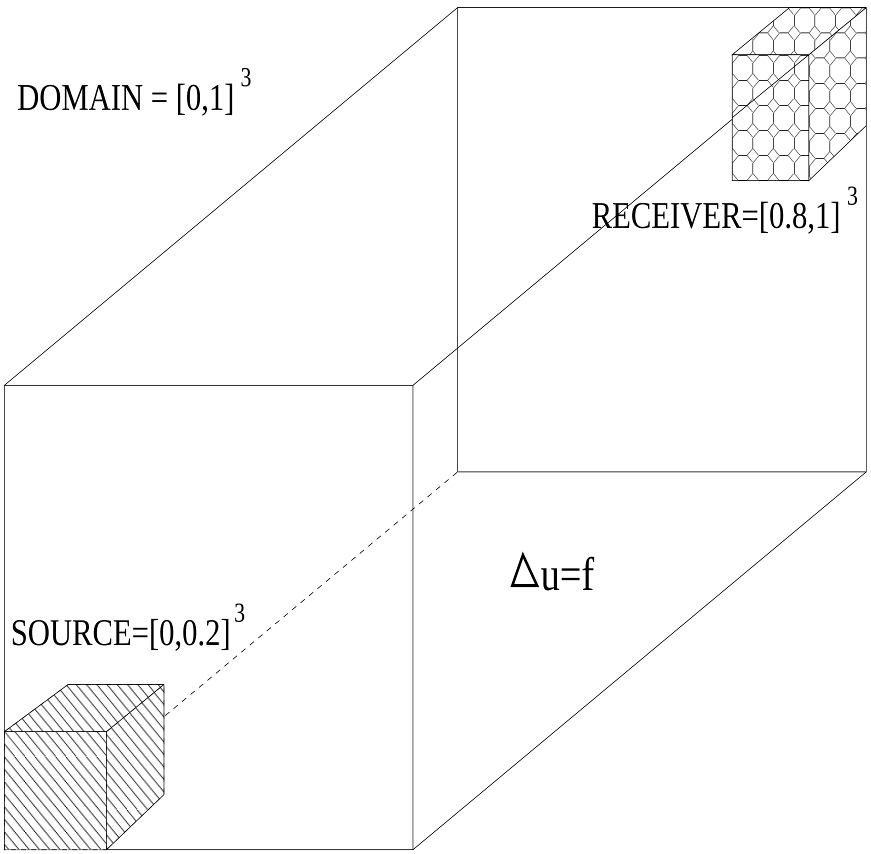
$$\frac{\| e^{(n+1)} \|_A}{\| e^{(0)} \|_A} \leq 0.01$$

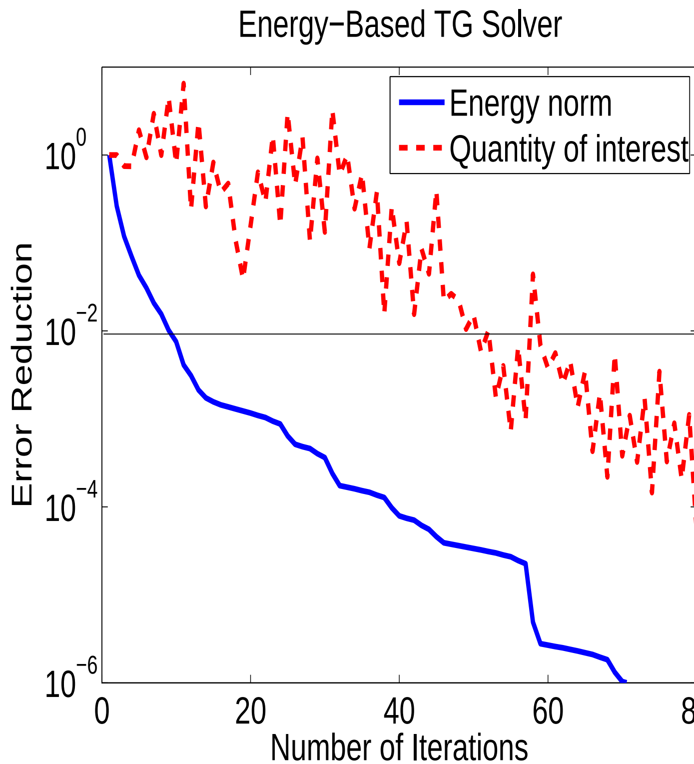
(Stopping Criteria)

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Motivation

MODEL PROBLEM I



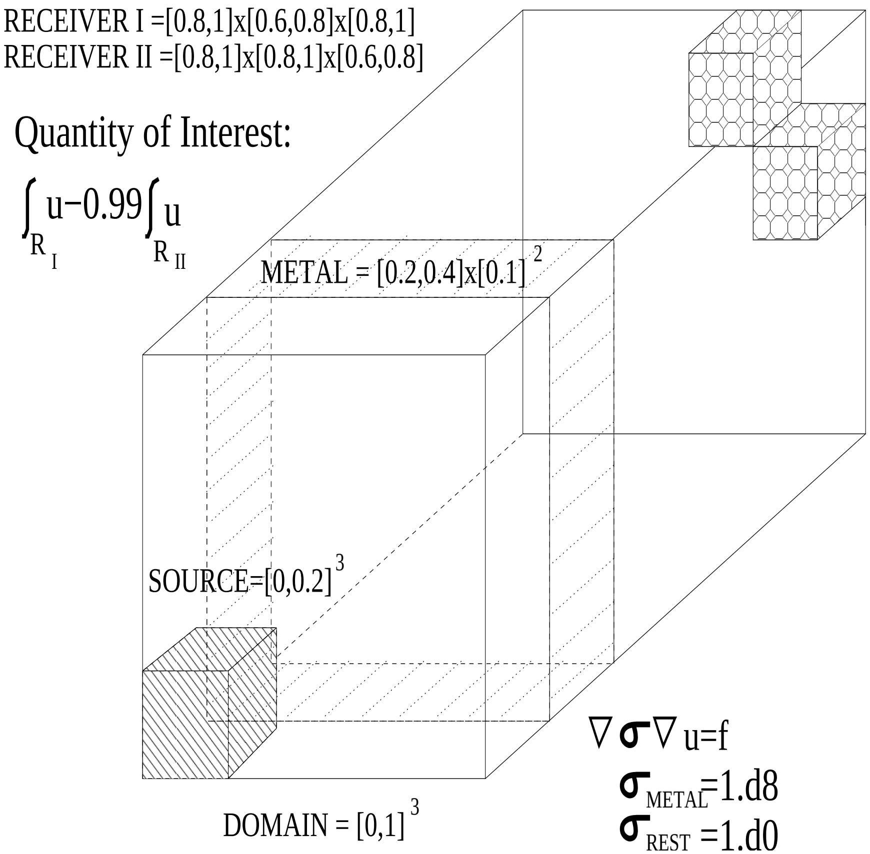


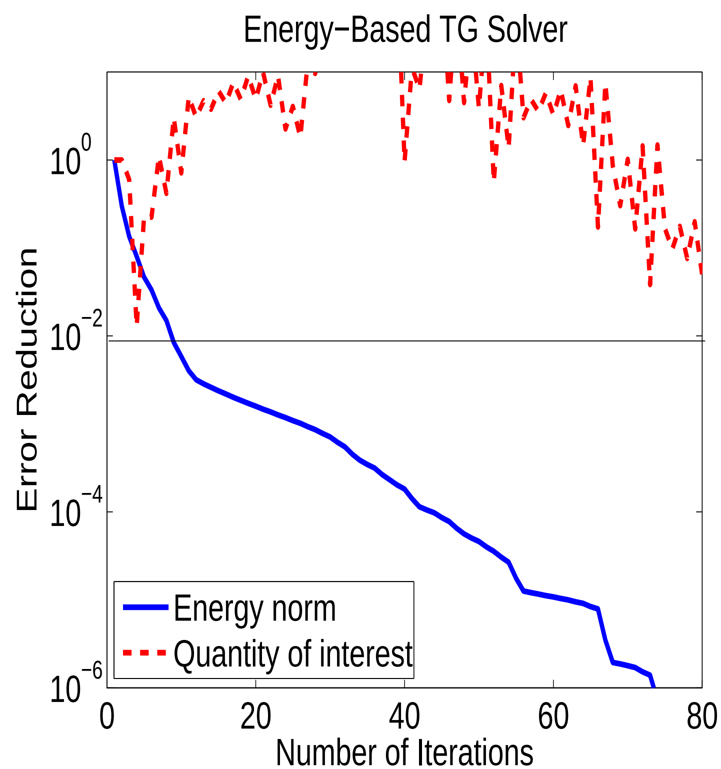
We need 50 iterations to converge!!

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Motivation

MODEL PROBLEM II





We need more than 80 iterations to converge!!

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Formulation (Part I)

We seek $L(u)$ such that $Au = f$ and $AG = l$. Consider the following iterative scheme:

$$\boldsymbol{r}_u^{(n+1)} = [\boldsymbol{I} - \boldsymbol{\alpha}^{(n)}\boldsymbol{AS}]\boldsymbol{r}_u^{(n)} \quad ; \quad \boldsymbol{r}_G^{(n+1)} = [\boldsymbol{I} - \boldsymbol{\beta}^{(n)}\boldsymbol{AS}]\boldsymbol{r}_G^{(n)}$$
$$\boldsymbol{u}^{(n+1)} = \boldsymbol{u}^{(n)} + \boldsymbol{\alpha}^{(n)}\boldsymbol{S}\boldsymbol{r}_u^{(n)} \quad ; \quad \boldsymbol{G}^{(n+1)} = \boldsymbol{G}^{(n)} + \boldsymbol{\beta}^{(n)}\boldsymbol{S}\boldsymbol{r}_G^{(n)}$$

where S is a matrix, and $\alpha^{(n)}, \beta^{(n)}$ are relaxation parameters. Then, we define our two grid solver as:

1 iteration with $S = S_F = \sum A_i^{-1}$ $+$
1 iteration with $S = S_C = P_C A_C^{-1} R_C$

How do we select $\alpha^{(n)}, \beta^{(n)}$ to be *optimal*?

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Formulation (Part II)

Recall that: $|L(u)| = |b(u,G)| \leq \sum_K |b_K(u,G)| \leq \sum_K ||u||_K ||G||_K$ —

Selection I (Goal1):

$$\alpha^{(n)} = \beta^{(n)} = \arg\min_{\alpha} \sum_K |b_K(u^* - u_\alpha^{(n+1)}, G_\alpha^{(n+1)})|$$

Selection II (Goal2):

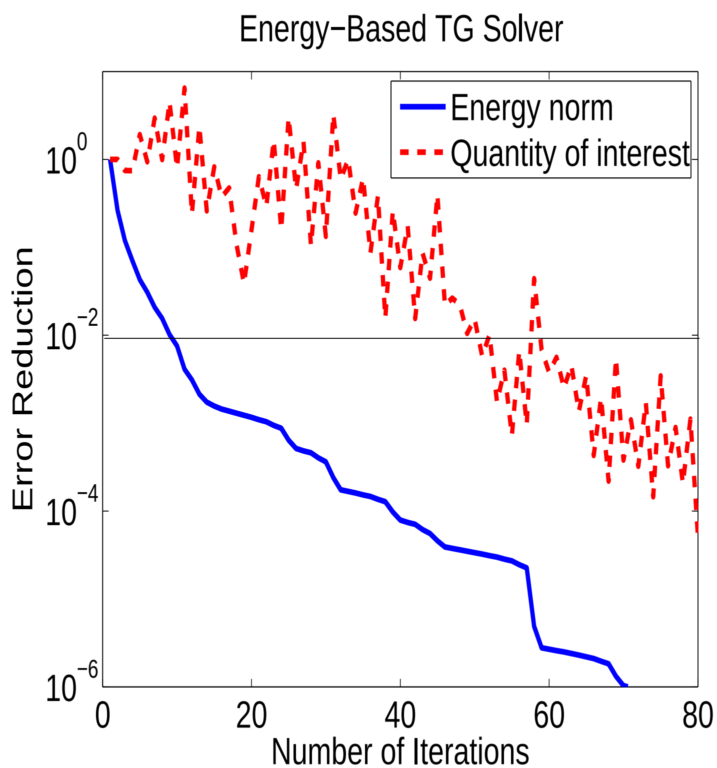
$$\alpha^{(n)} = \beta^{(n)} = \arg\min_{\alpha} \sum_K ||u^* - u_\alpha^{(n+1)}||_K \; ||G_\alpha^{(n+1)}||_K$$

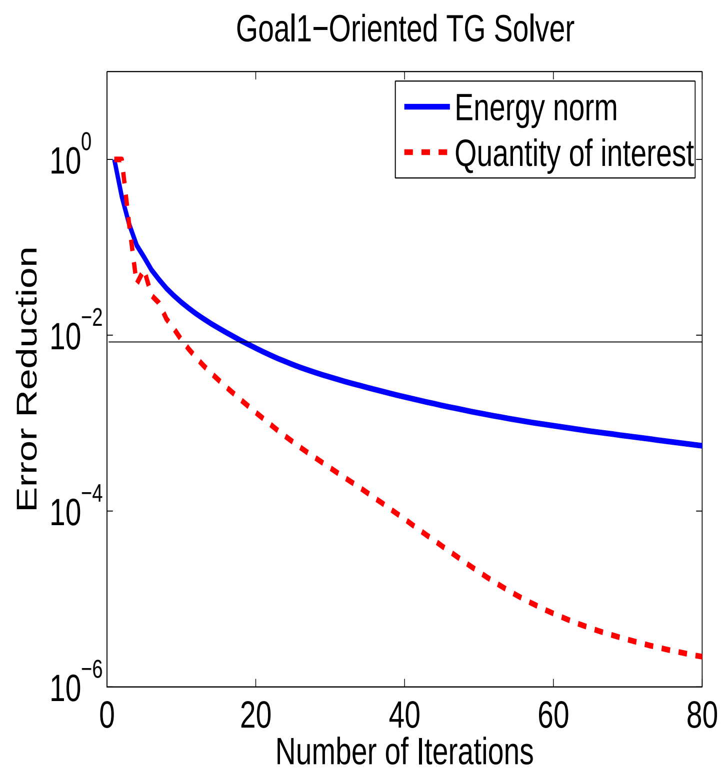
$u^* = u^{(n)} + (S - SAS)r^{(n)}$

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Numerical Results

MODEL PROBLEM I





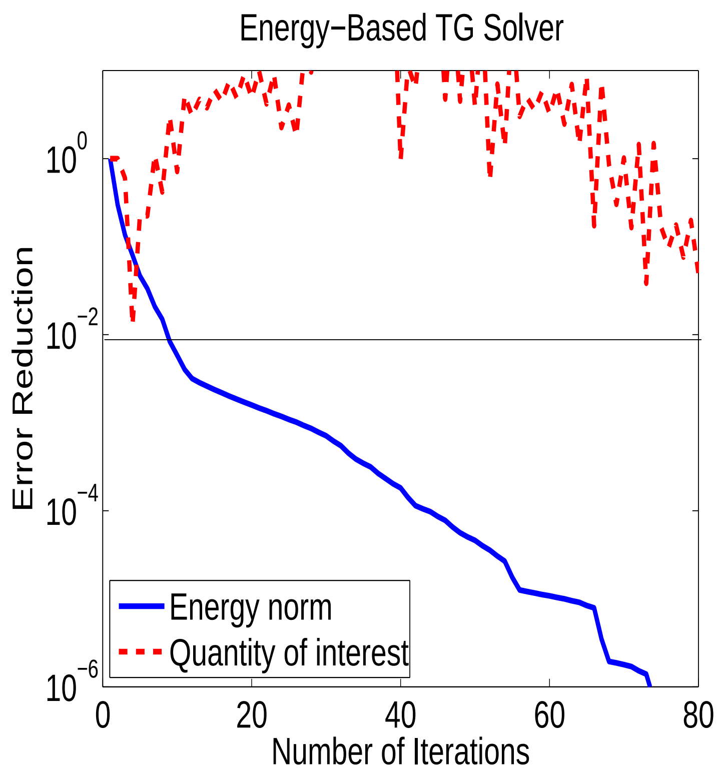
50 iter. to converge **8 iter. to converge**

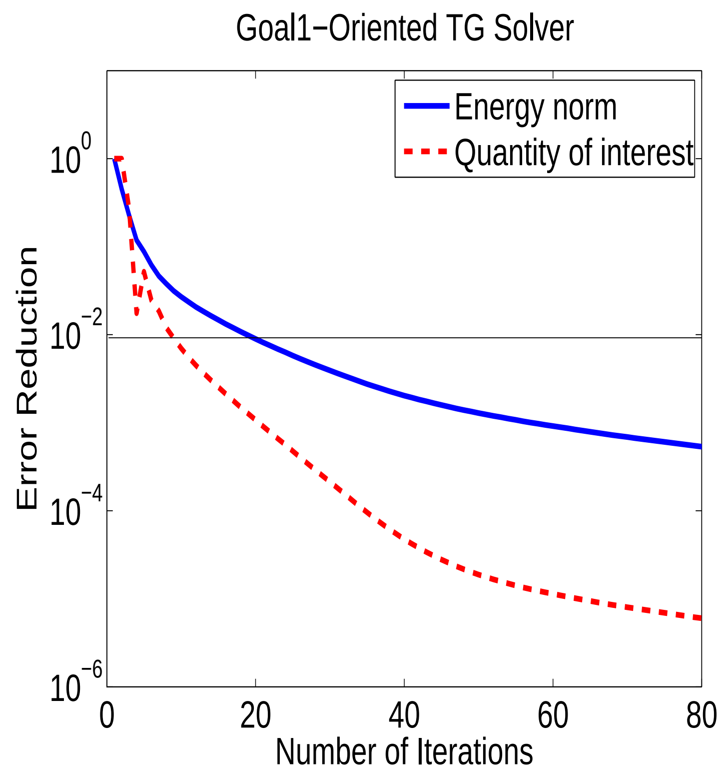
With the GOAL-ORIENTED solver we reduce the number of iterations

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Numerical Results

MODEL PROBLEM II





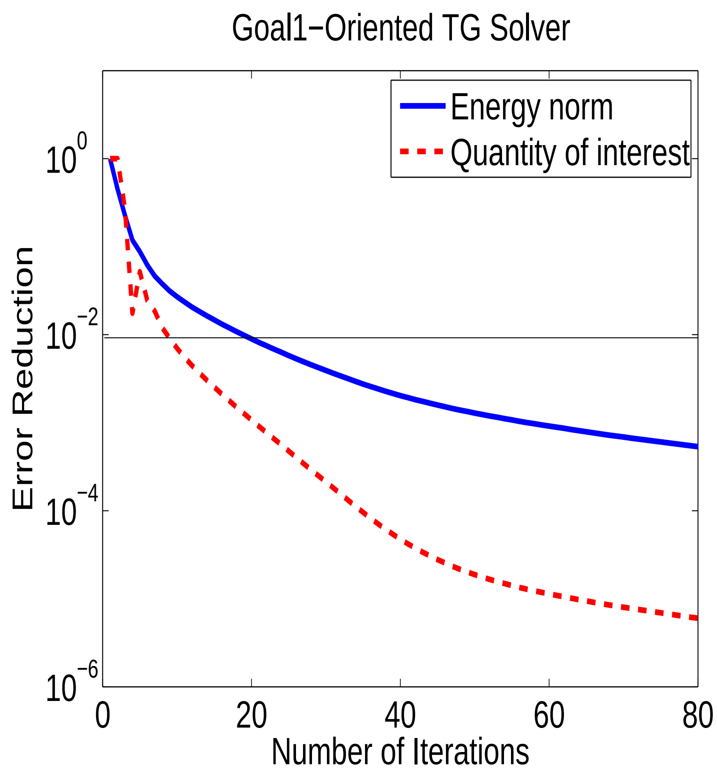
$>$80 iter. to converge **8 iter. to converge**

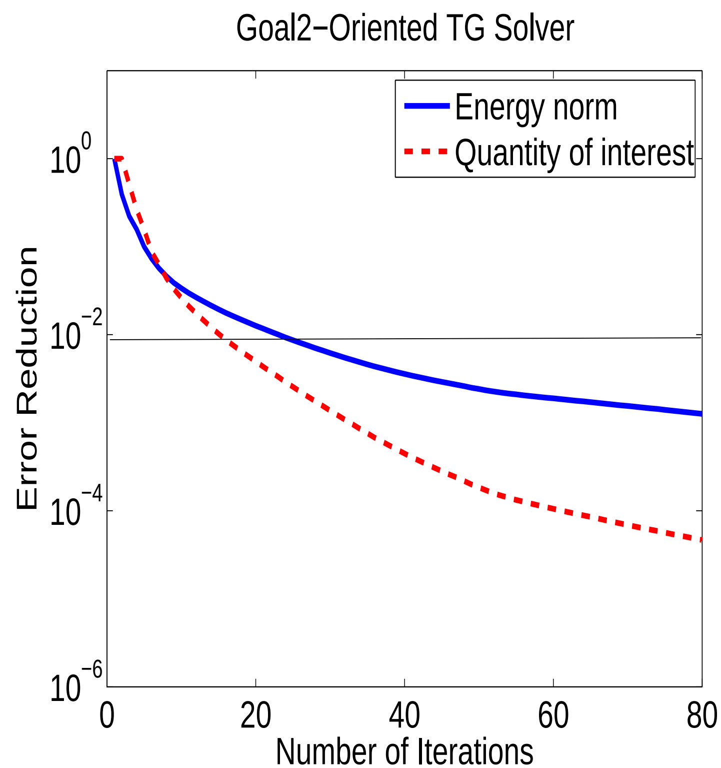
We only converge with the GOAL-ORIENTED solver

TG SOLVER: GOAL-ORIENTED

Goal-Oriented Solver: Numerical Results

MODEL PROBLEM II





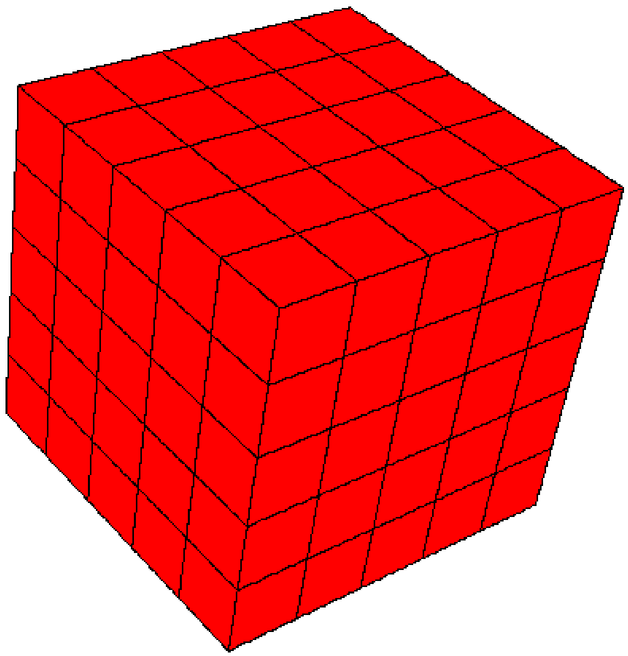
8 iter. to converge **16 iter. to converge**

Goal1 algorithm converges in less iterations than Goal2

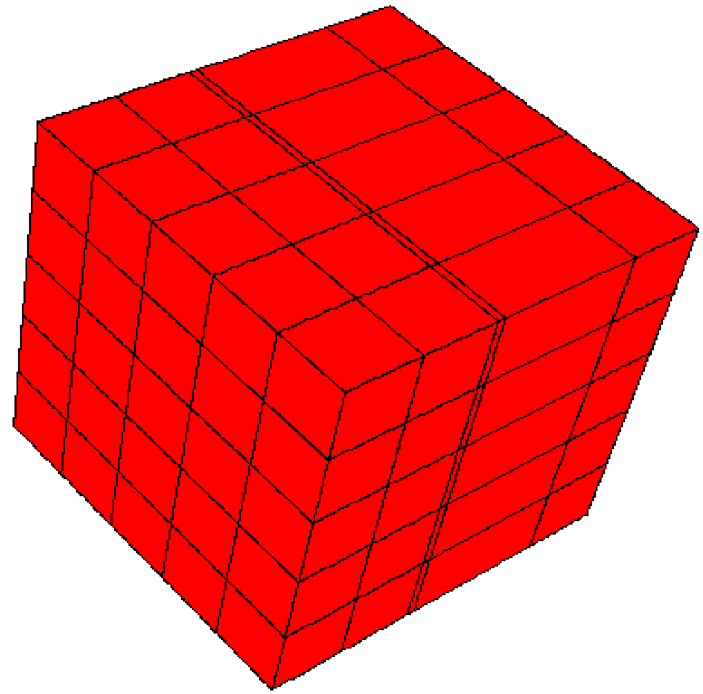
TG SOLVER: ELONGATED ELEMENTS

Elongated Elements

MODEL PROBLEM I (Initial Grid)



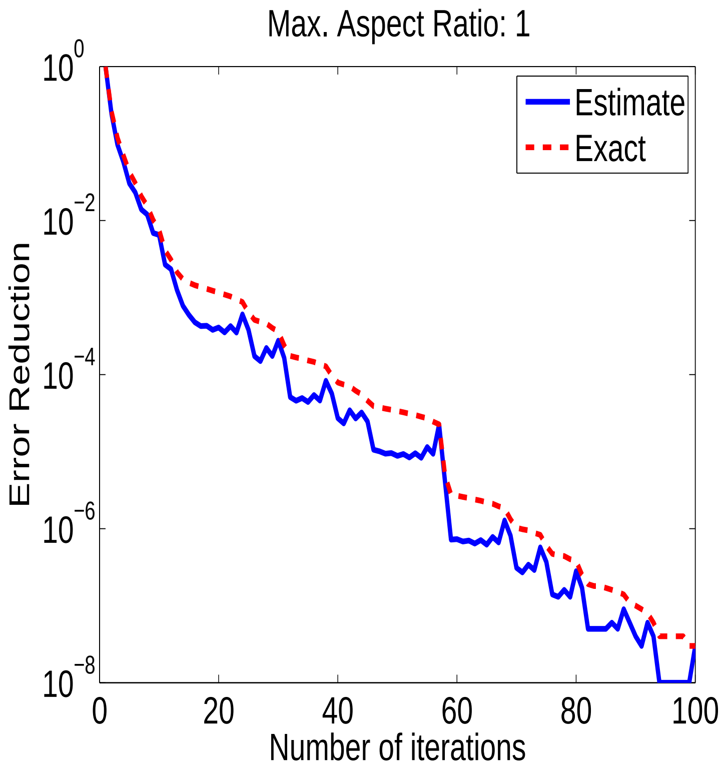
Isotropic

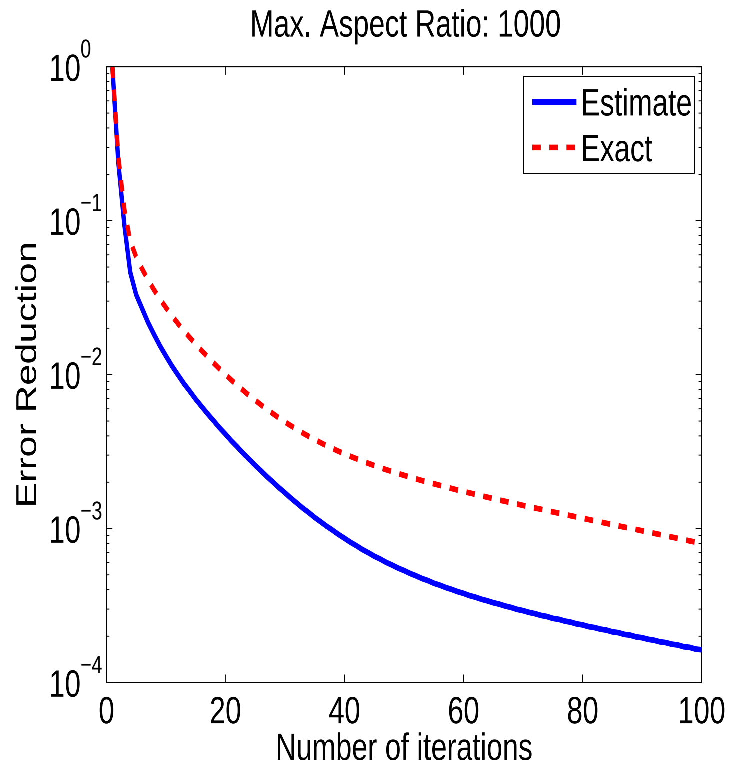


Anisotropic

TG SOLVER: ELONGATED ELEMENTS

Elongated Elements

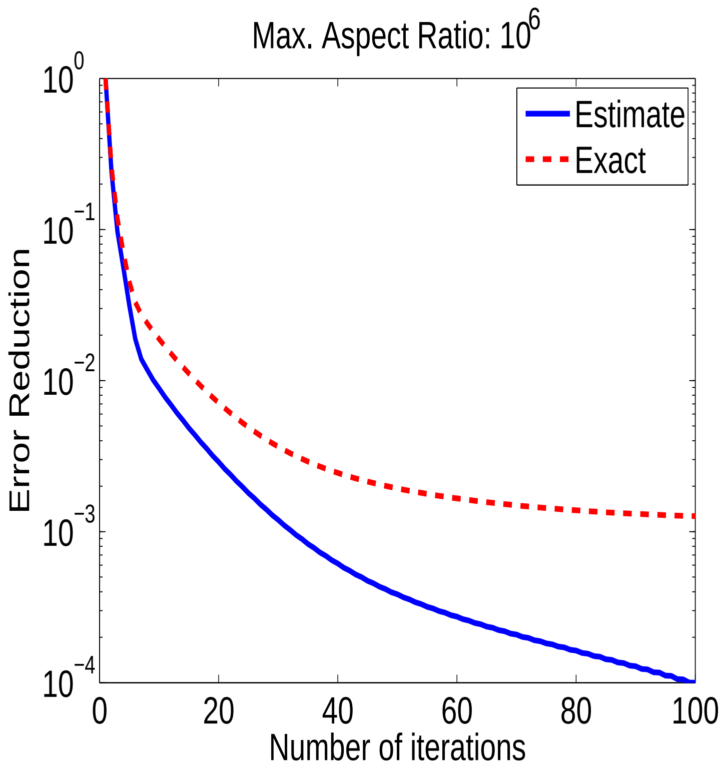


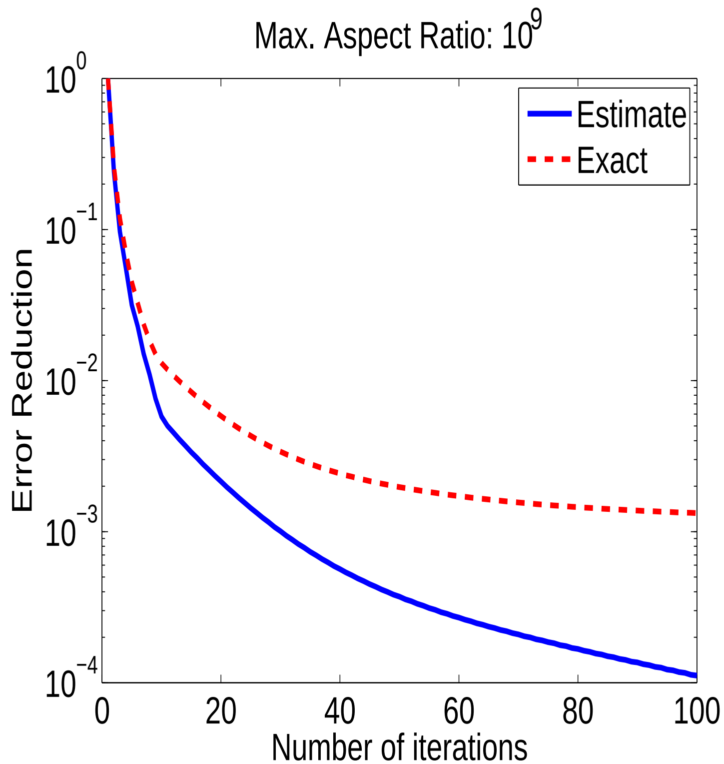


For elongated elements, convergence and error estimation degenerates

TG SOLVER: ELONGATED ELEMENTS

Elongated Elements

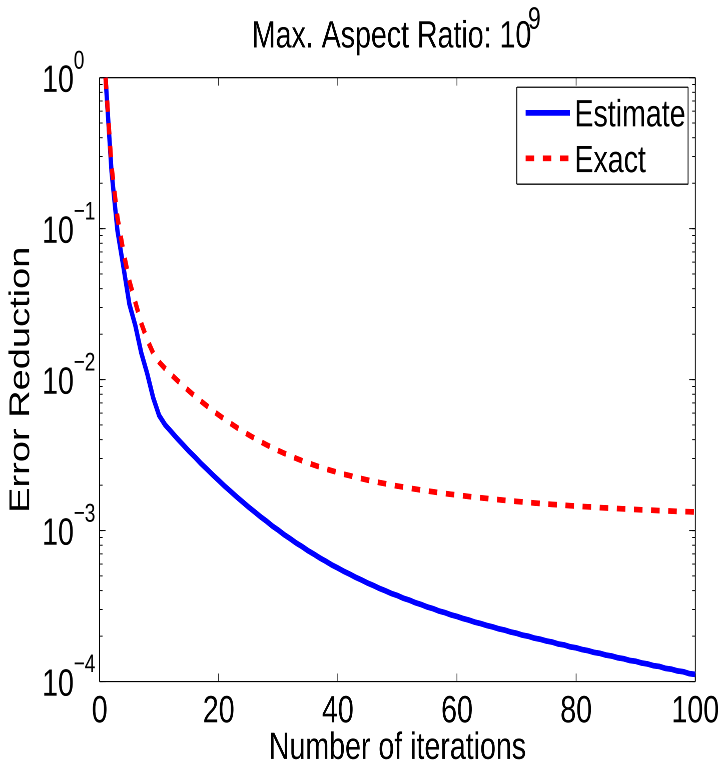


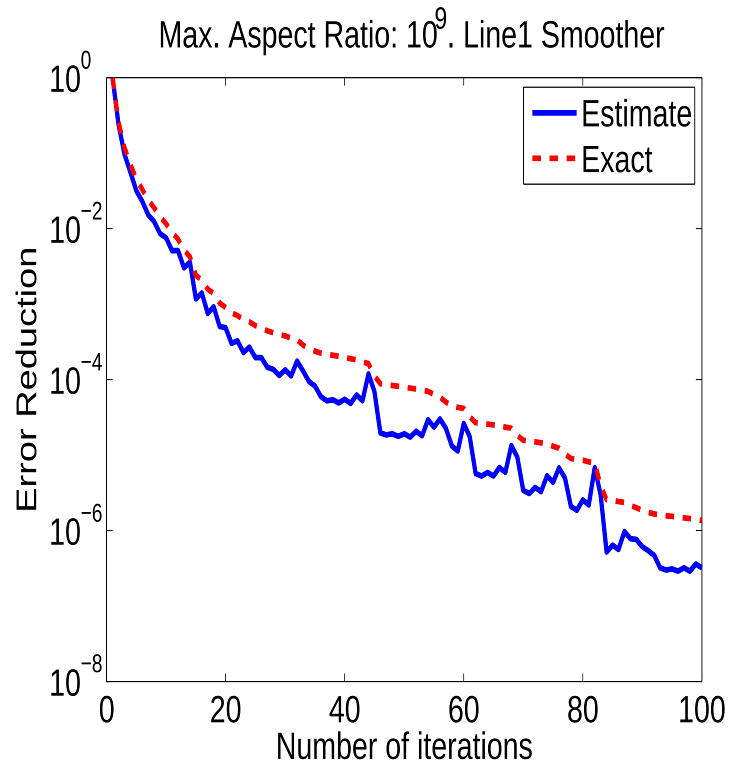


For elongated elements, convergence and error estimation degenerates

TG SOLVER: ELONGATED ELEMENTS

Elongated Elements

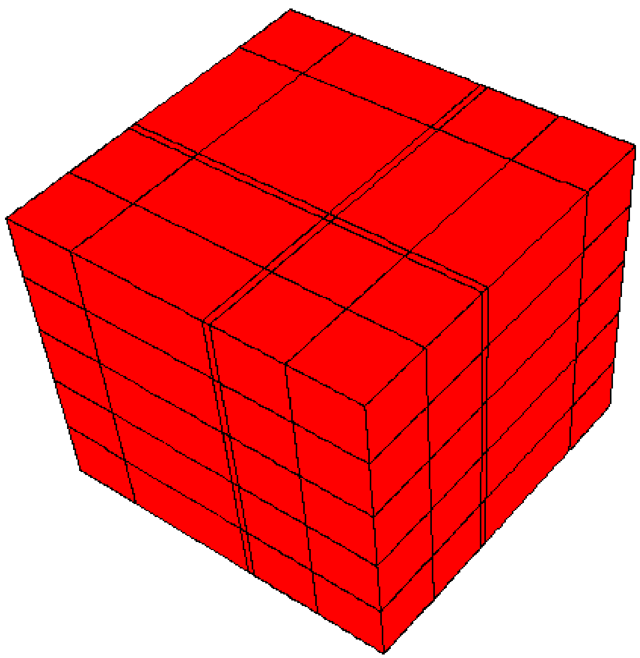


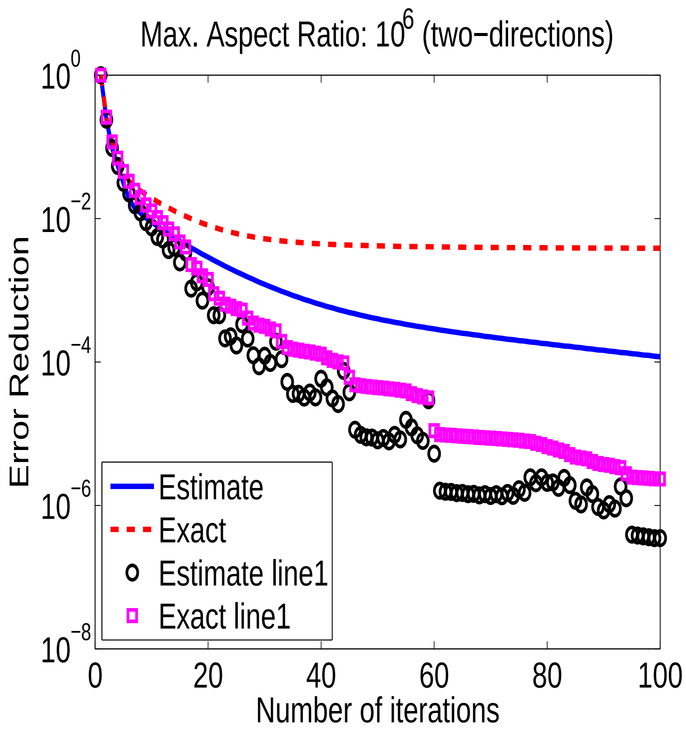


Line1 Smoother=old smoother + additional block composed of all d.o.f. associated to elongated elements

TG SOLVER: ELONGATED ELEMENTS

Elongated Elements

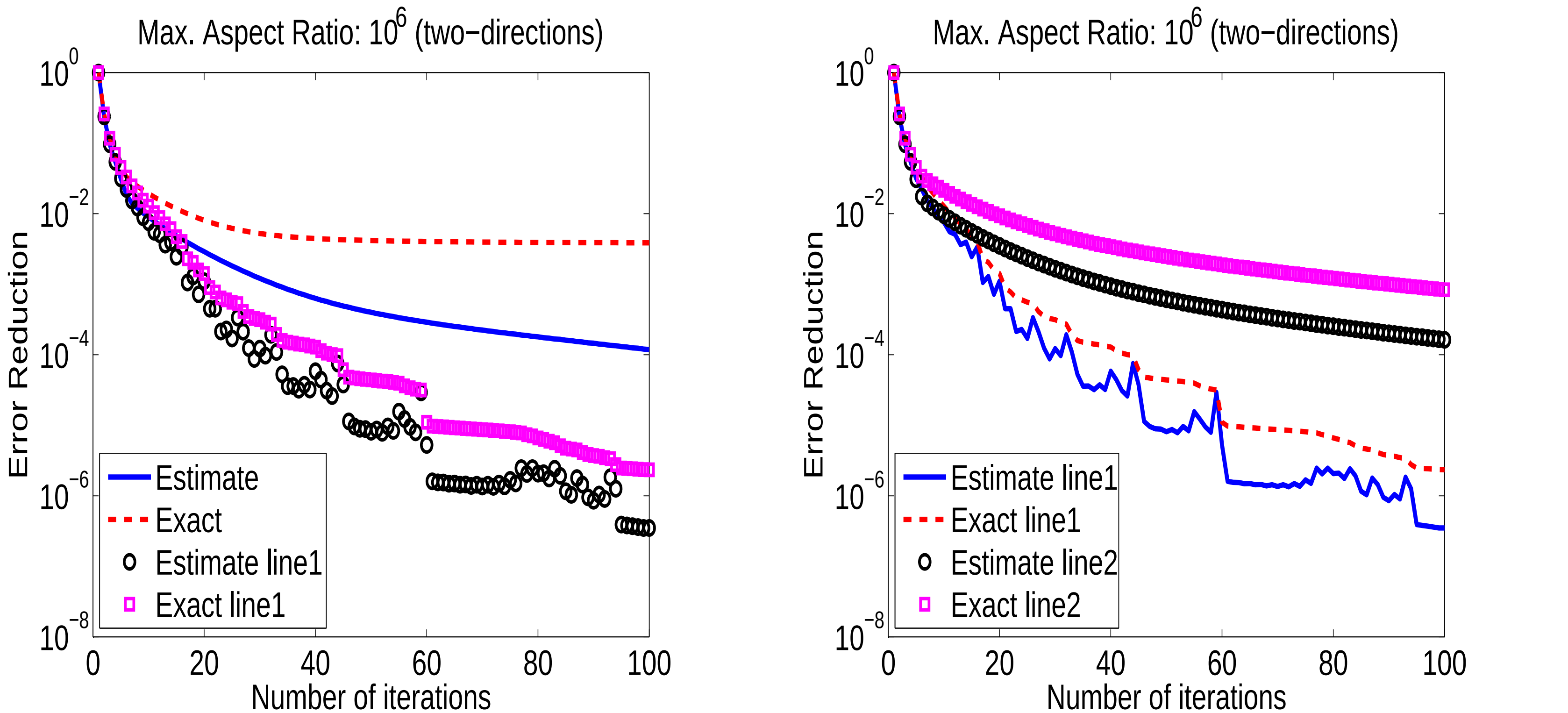




Line smoothers are necessary in presence of elongated elements

TG SOLVER: ELONGATED ELEMENTS

Elongated Elements



Line2 Smoother=old smoother + additional block composed of all EDGE and VERTEX d.o.f. associated to elongated elements

TG SOLVER: IMPLEMENTATION

Implementation Details

Block-Jacobi Smoother: PATCH-LEVEL (UNASSEMBLED) OPERATIONS.

- **Enables flexible (adaptive) smoother selection.**
- **Does not require inversion (only LU factor).**

Stiffness matrix: ASSEMBLED.

- **Facilitates flexible (adaptive) smoother selection.**
- **Minimizes storage (avoids node repetition).**

Transfer operators: ACTING ON RIGHT HAND SIDE, ASSEMBLED.

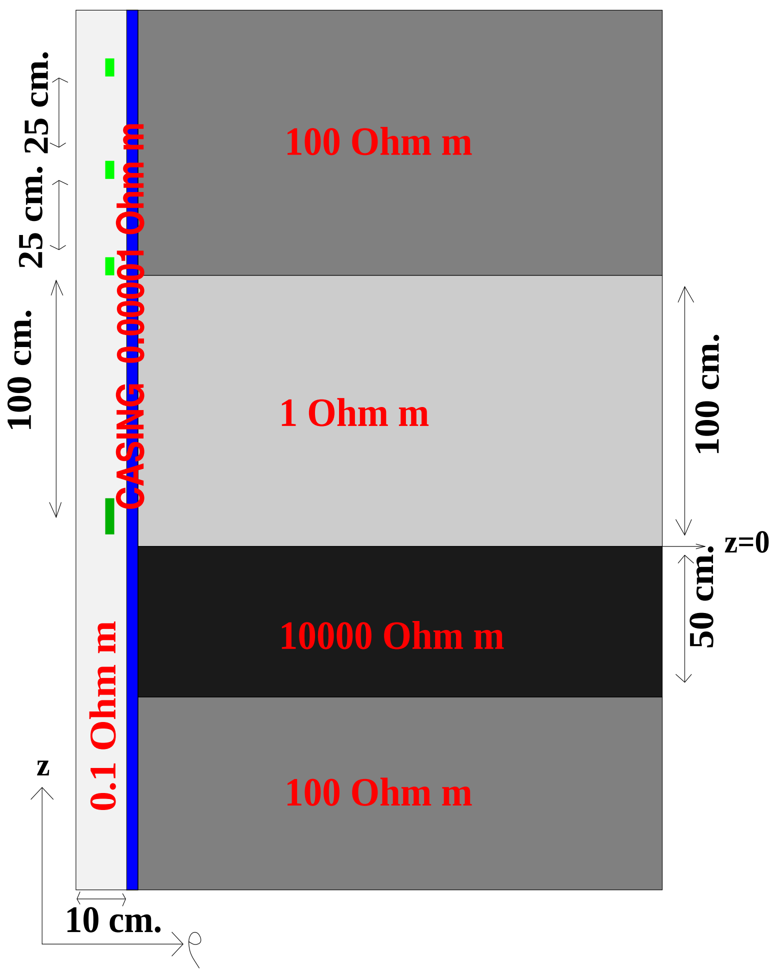
- **Avoids using matrix-matrix multiplications.**
- **Logic consistent with that of the stiffness matrix.**

Coarse-grid solve: ASSEMBLED.

- **Logic consistent with that of the stiffness matrix.**
- **Logic consistent with that of the smoother.**

NUMERICAL RESULTS

Model Problem with Steel Casing



Frequency: 0 Hz.

Casing resistivity: 10^{-5} Ohm $\cdot$ m.

Casing width: 0.01127 m

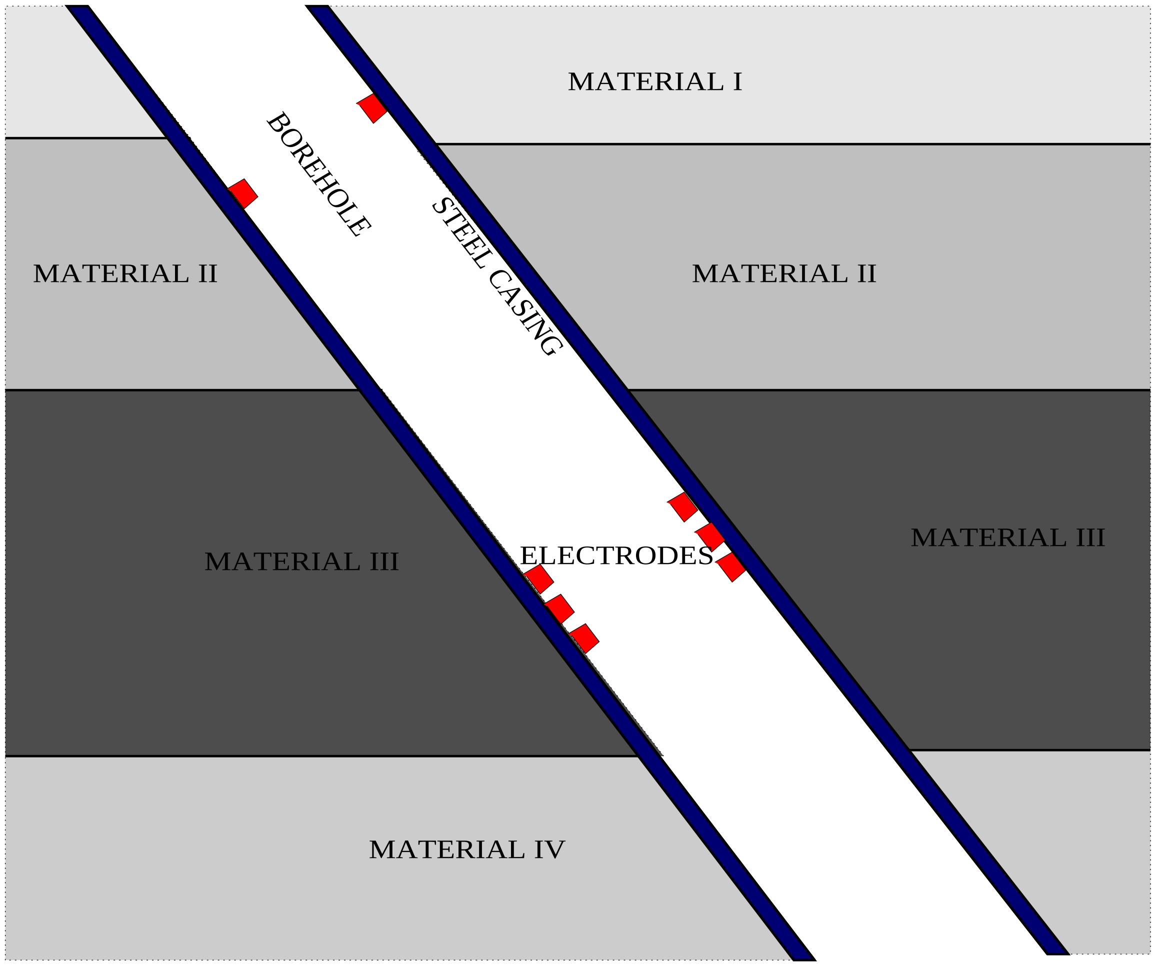
Large constrast in resistivity

Electrodes

Size (domain): 1000m x 2000m

NUMERICAL RESULTS

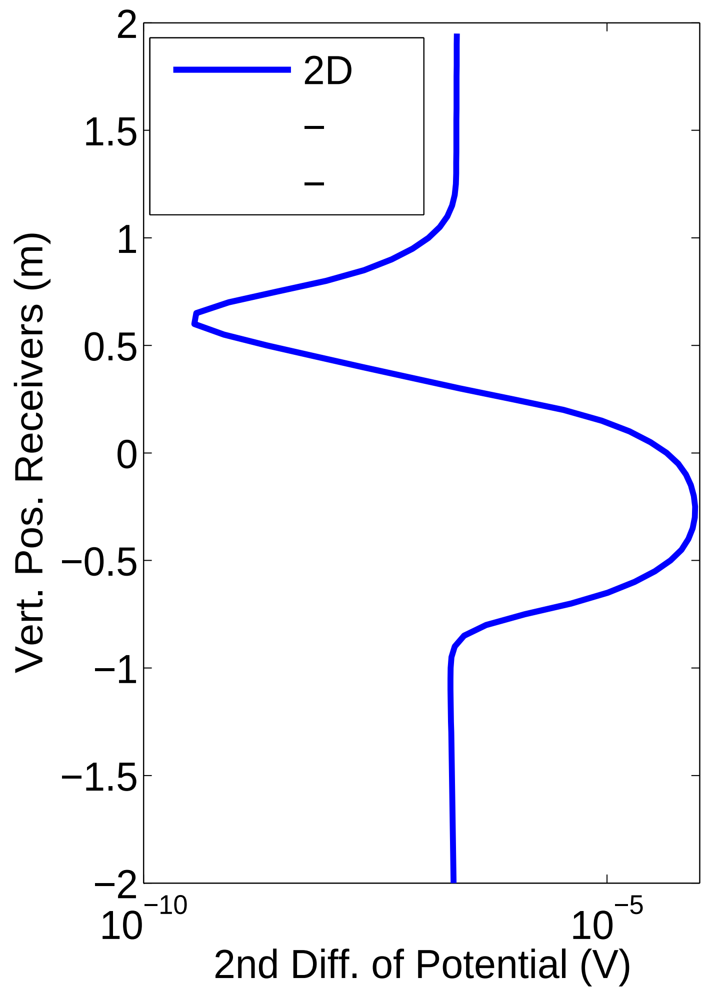
Deviated Cased Wells



Objective: Determine 2nd difference of potential at the receiver electrodes.

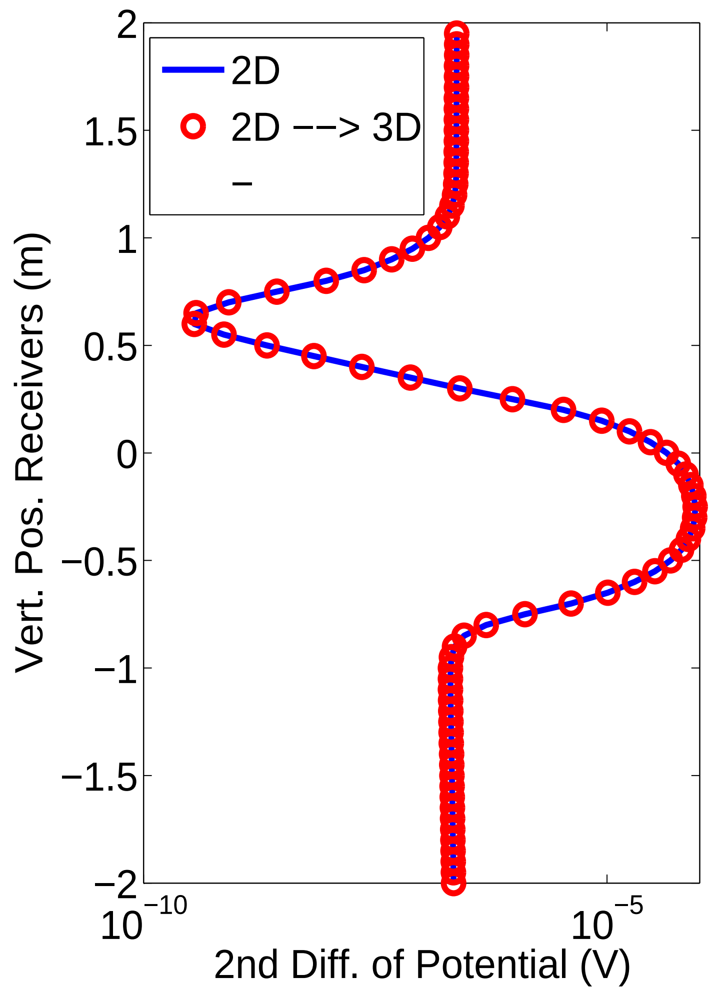
NUMERICAL RESULTS

Axisymmetric problem



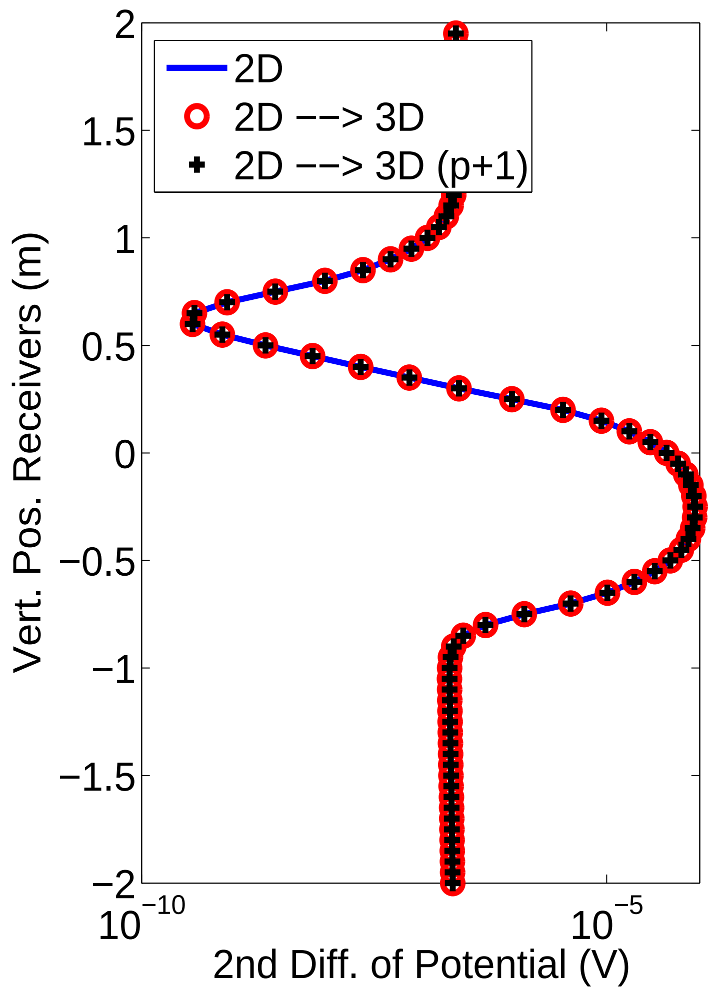
NUMERICAL RESULTS

Axisymmetric problem



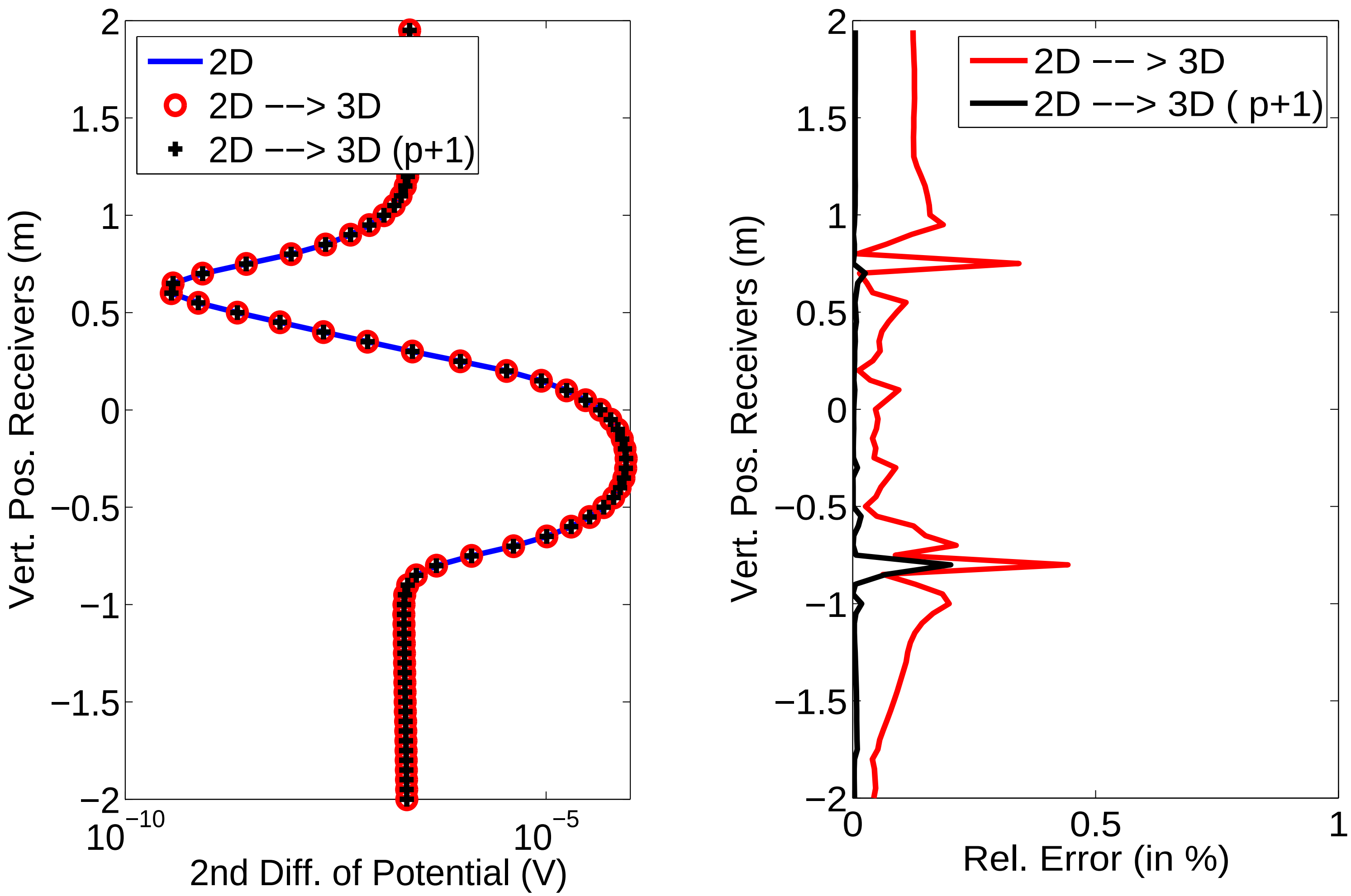
NUMERICAL RESULTS

Axisymmetric problem



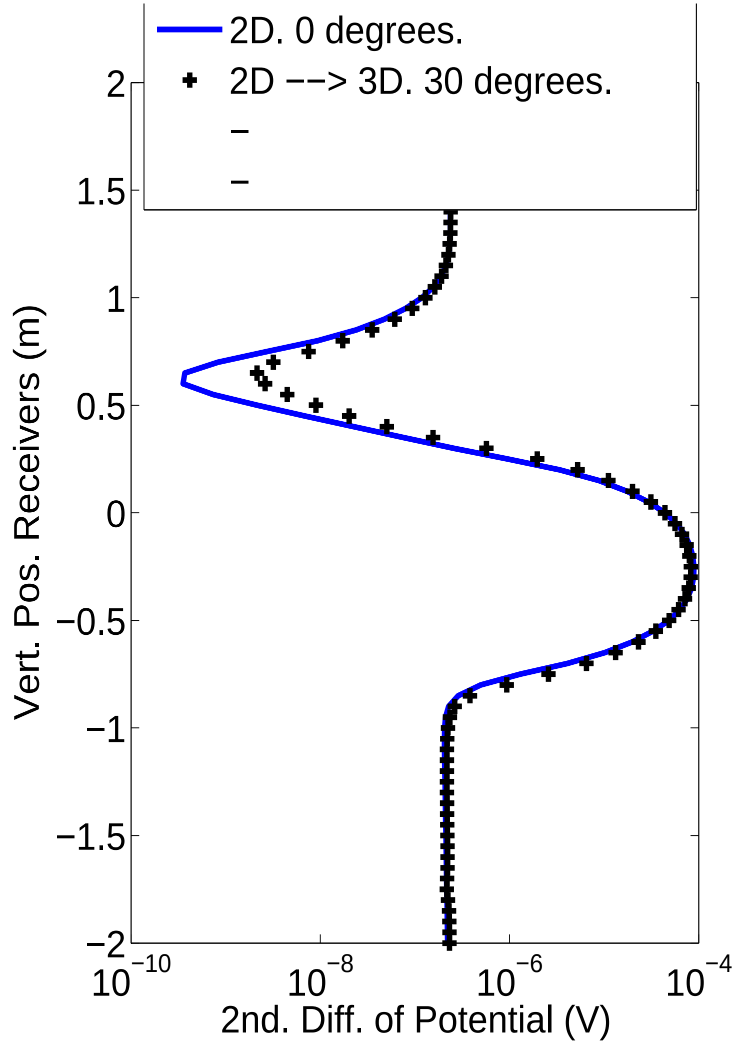
NUMERICAL RESULTS

Axisymmetric problem



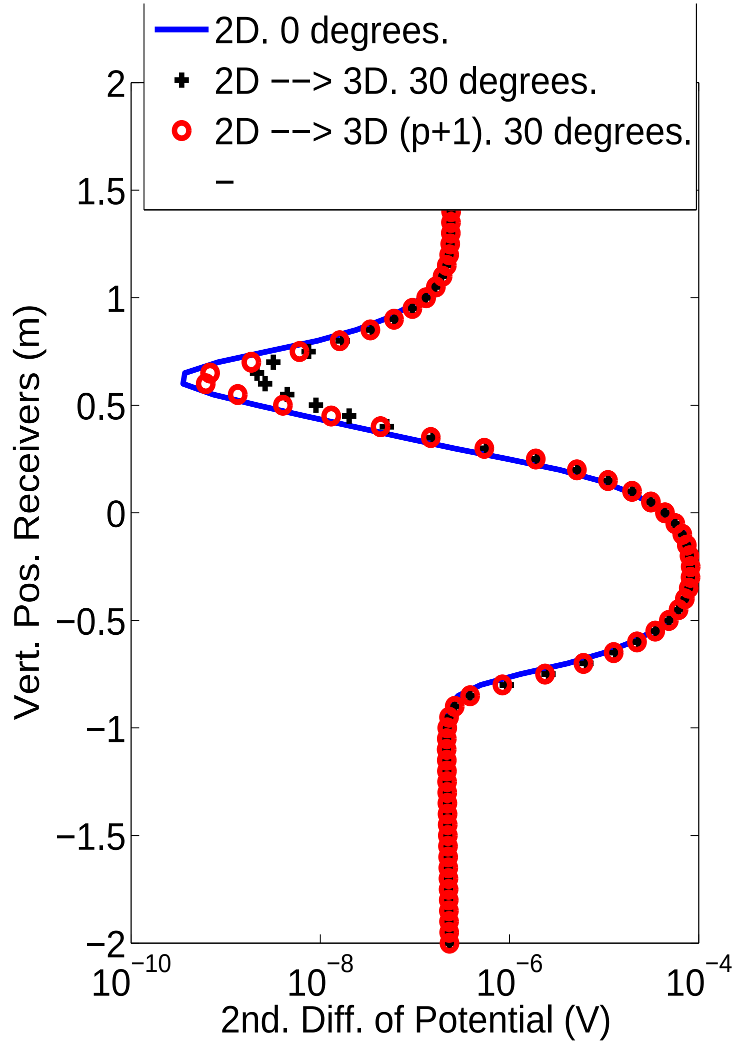
NUMERICAL RESULTS

30 degrees deviated well



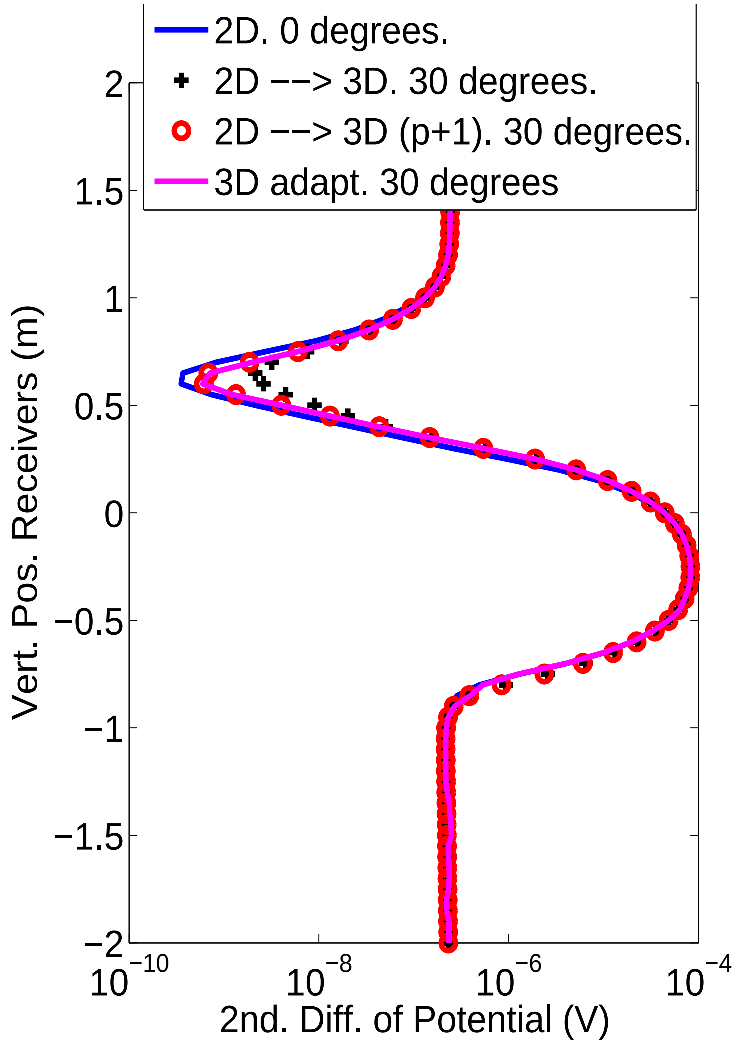
NUMERICAL RESULTS

30 degrees deviated well



NUMERICAL RESULTS

30 degrees deviated well



NUMERICAL RESULTS

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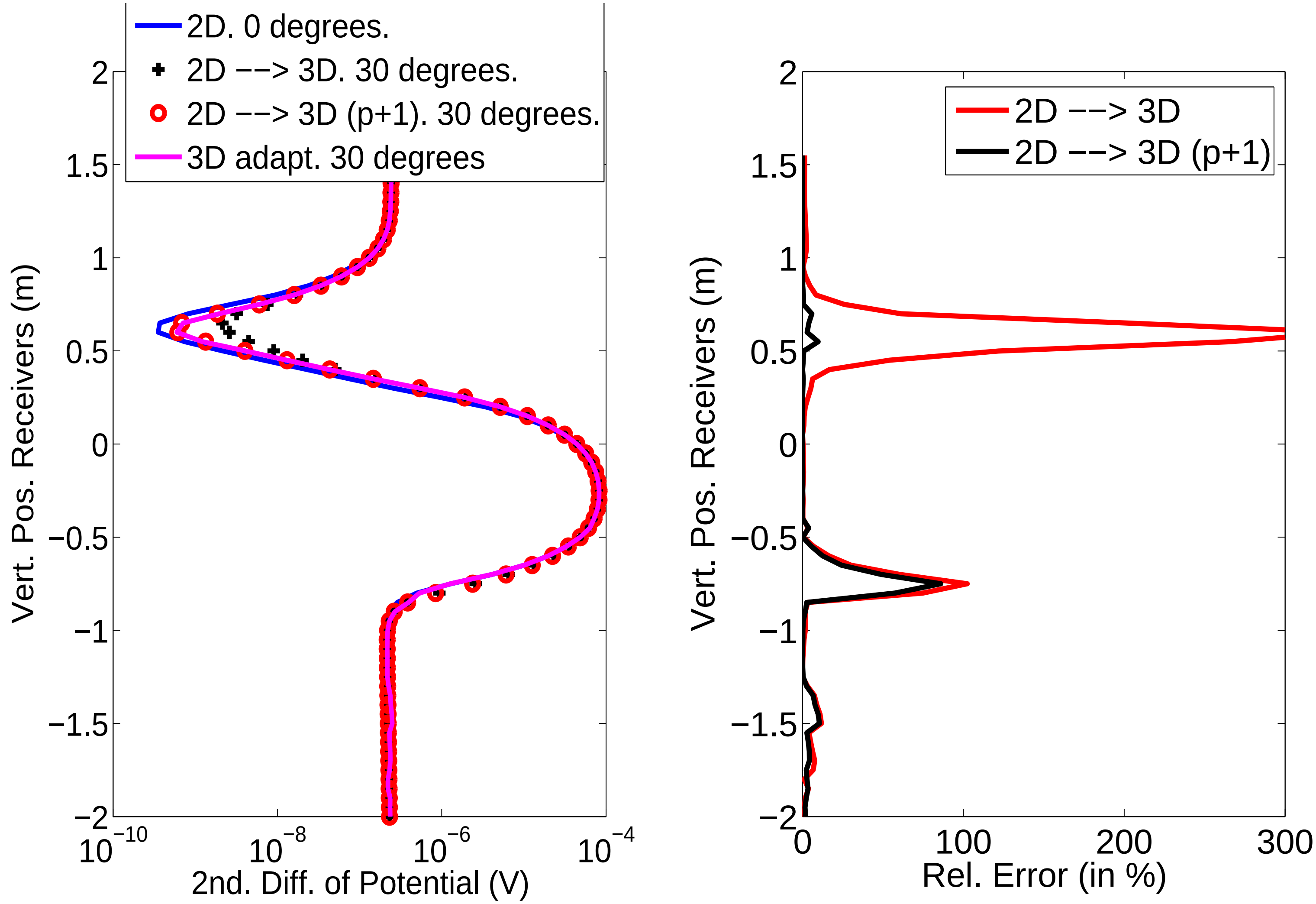
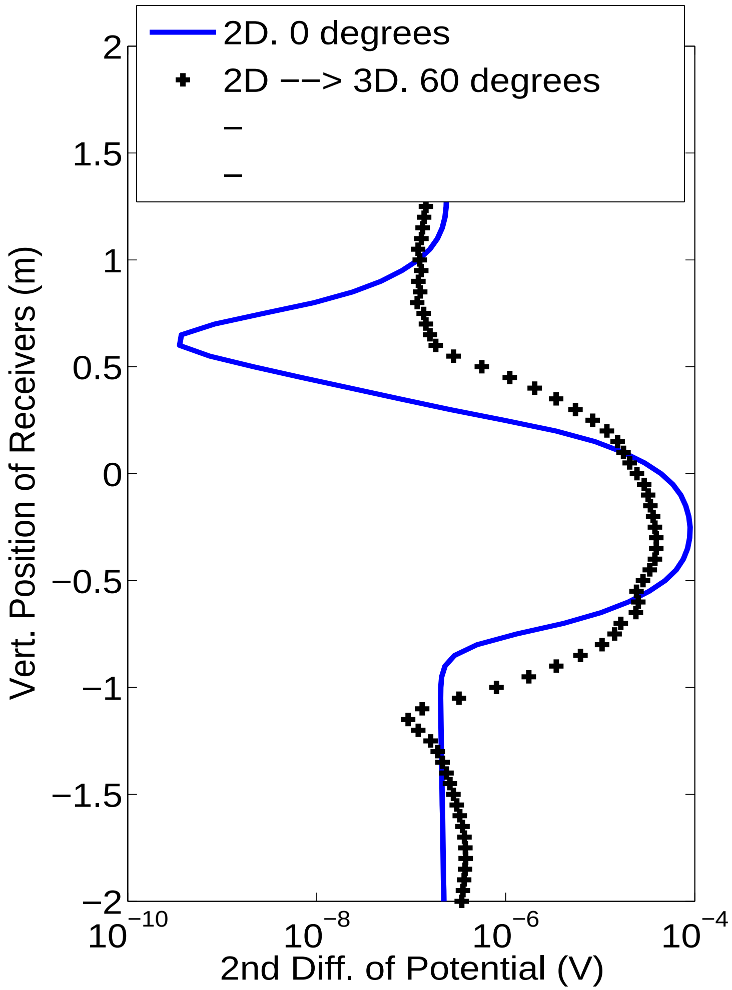


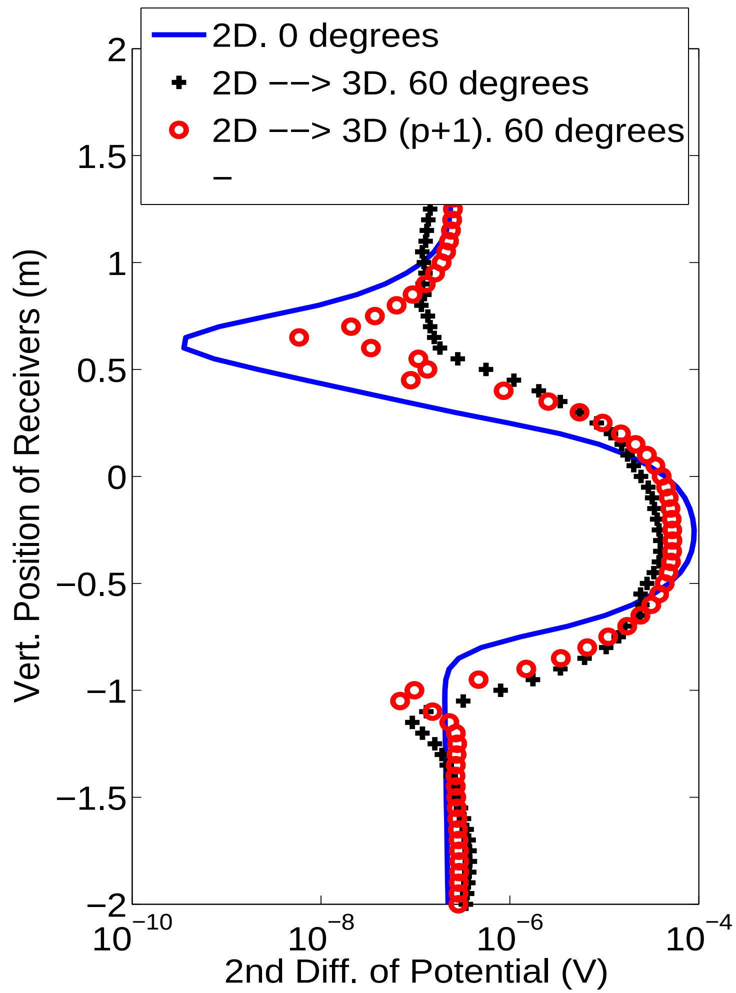
NUMERICAL RESULTS

60 degrees deviated well



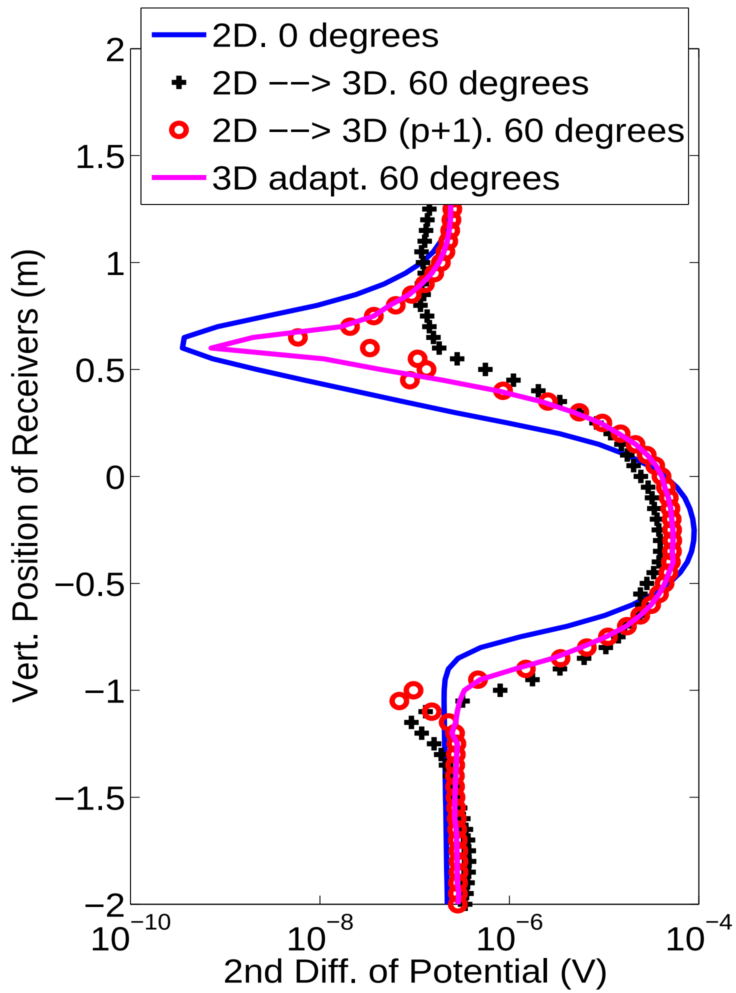
NUMERICAL RESULTS

60 degrees deviated well



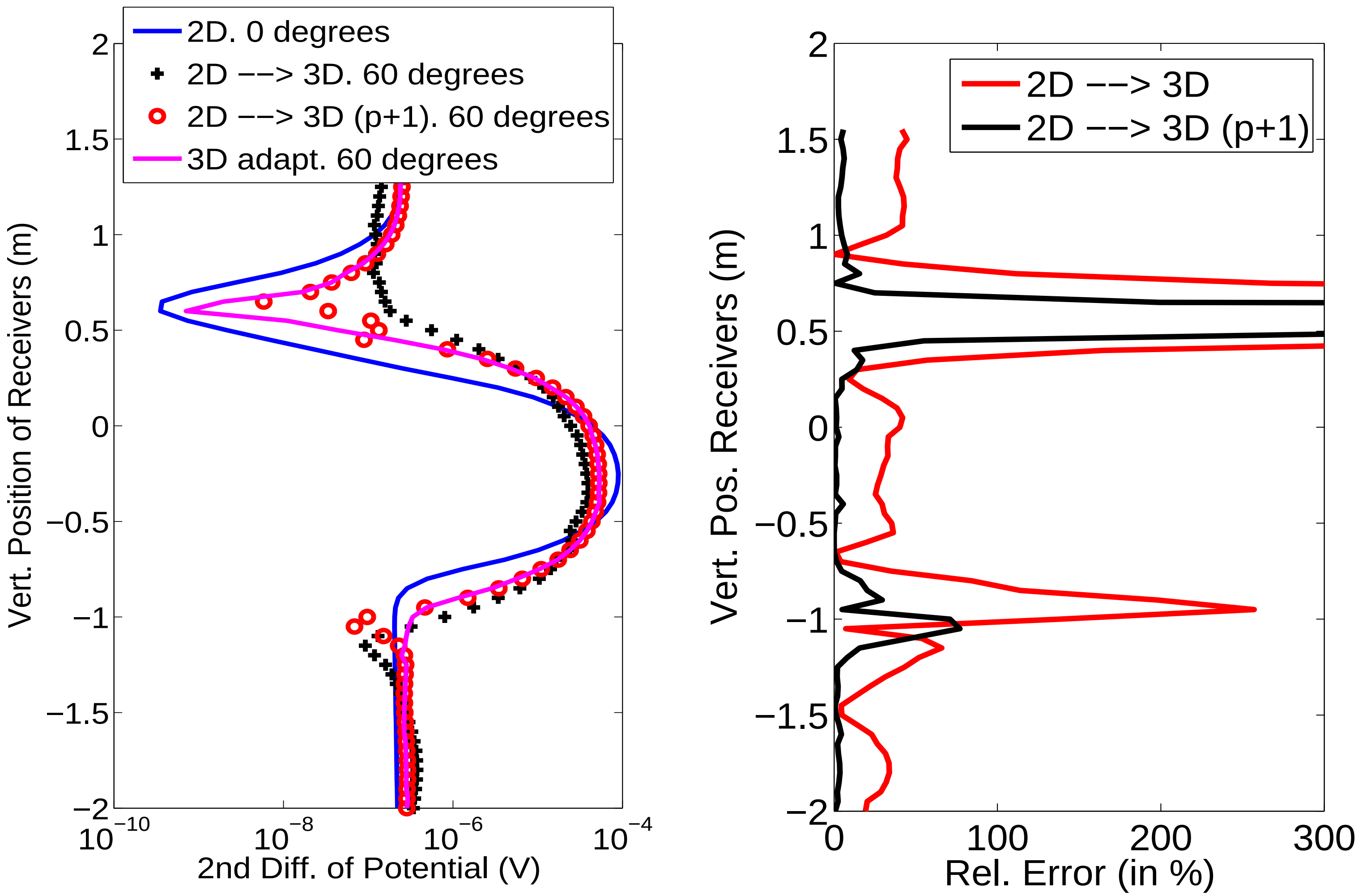
NUMERICAL RESULTS

60 degrees deviated well



NUMERICAL RESULTS

60 degrees deviated well



CONCLUSIONS AND FUTURE WORK

CONCLUSIONS

- A goal-oriented adaptive strategy should be accompanied by a goal-oriented iterative solver.
- Error estimation may fail in presence of elongated elements.
- Elongated elements should be identified before solving the actual problem.
- Line smoothers are needed to converge in presence of elongated elements.

FUTURE WORK

- New error estimators.
- Electrodynamic problems.

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MARATHON Marathon Oil Corporation

HYDRO

OXY

BR PETROBRAS

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