Integration of hp-adaptivity with a Two Grid Solver: Applications to Electromagnetics.

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OVERVIEW

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5. A Two Grid Solver for Symmetric and Positive Definite Problems.
6. A Two Grid Solver for Electromagnetics.
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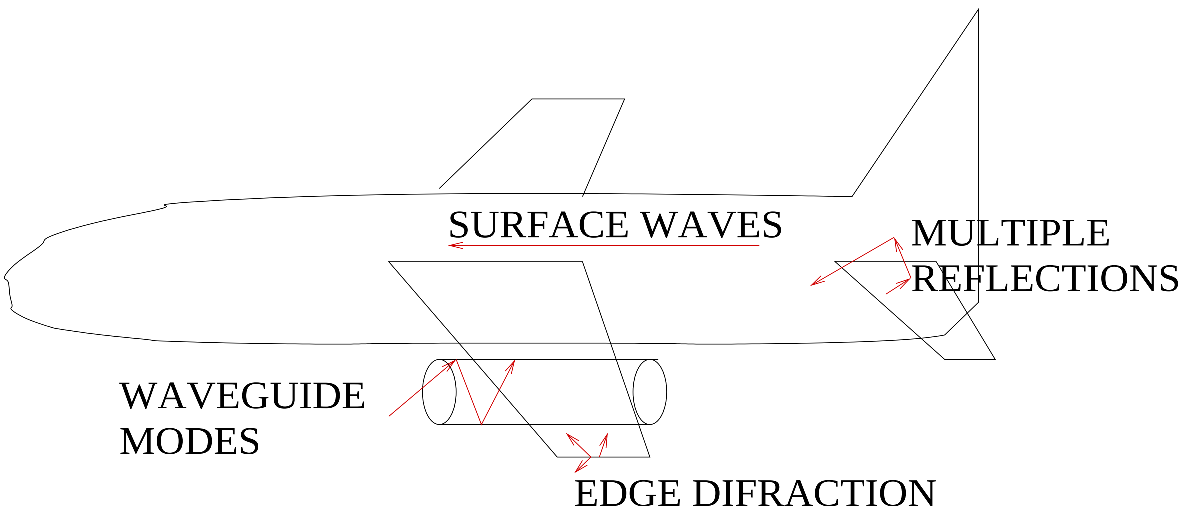
MOTIVATION

Radar Cross Section (RCS) Analysis



RCS$=4\pi\frac{\text{Power scattered to receiver per unit solid angle}}{\text{Incident power density}} = \lim_{r\to\infty} 4\pi r^2 \frac{|\boldsymbol{E}^s|}{|\boldsymbol{E}^i|}$.

Goal: Determine the RCS of a plane.

MAXWELL'S EQUATIONS

Time Harmonic Maxwell's Equations:

$$\nabla \times \mathbf{E} = -j\mu\omega\mathbf{H}$$
$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \sigma\mathbf{E} + \mathbf{J}^{imp}$$

Reduced Wave Equation:

$$\boldsymbol{\nabla} \times \left(\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}\right) - (\omega^2\epsilon - j\omega\sigma)\boldsymbol{E} = -j\omega\boldsymbol{J}^{imp} \, ,$$

Boundary Conditions (BC):

- **Dirichlet BC at a PEC surface:**

$$\mathbf{n} \times \boldsymbol{E}^s = -\mathbf{n} \times \boldsymbol{E}^{inc} \qquad \mathbf{n} \times \boldsymbol{E} = 0$$

- **Neumann continuity BC at a material interface:**

$$\mathbf{n} \times \frac{1}{\mu}\nabla \times \boldsymbol{E}^s = -\mathbf{n} \times \frac{1}{\mu}\nabla \times \boldsymbol{E}^{inc} \qquad \mathbf{n} \times \frac{1}{\mu}\nabla \times \boldsymbol{E} = -j\omega\mathbf{J}_S^{imp}$$

- **Silver Müller radiation condition at ∞:**

$$\mathbf{e}_r \times (\nabla \times \boldsymbol{E}^s) - jk_0 \times \boldsymbol{E}^s = O(r^{-2})$$

MAXWELL'S EQUATIONS for hp-FEM

Variational formulation

The reduced wave equation in Ω,

$$\nabla \times \left(\frac{1}{\mu} \nabla \times E \right) - (\omega^2 \epsilon - j\omega\sigma) E = -j\omega J^{imp} \,,$$

A variational formulation

$$\left\{ \begin{array}{l} \text{Find } \mathrm{E} \in H_D(\mathrm{curl};\Omega) \text{ such that} \\ \displaystyle\int_\Omega \frac{1}{\mu} (\nabla \times \mathrm{E}) \cdot (\nabla \times \bar{F}) dx - \int_\Omega (\omega^2 \epsilon - j\omega\sigma) E \cdot \bar{\mathrm{F}} dx = \\ \qquad\qquad \displaystyle -j\omega \left\{ \int_\Omega J^{imp} \cdot \bar{F} dx + \int_{\Gamma_2} J_S^{imp} \cdot \bar{F} dS \right\} \quad \text{for all } F \in H_D(\mathrm{curl};\Omega) \,. \end{array} \right.$$

A regularized variational formulation (using *Lagrange multipliers*):

$$\left\{ \begin{array}{l} \text{Find } E \in H_D(\mathrm{curl};\Omega), p \in H_D^1(\Omega) \text{ such that} \\ \displaystyle\int_\Omega \frac{1}{\mu} (\nabla \times E)(\nabla \times \bar{F}) dx - \int_\Omega (\omega^2 \epsilon - j\omega\sigma) E \cdot \bar{F} dx - \int_\Omega (\omega^2 \epsilon - j\omega\sigma) \nabla p \cdot \bar{F} dx = \\ \qquad\qquad \displaystyle -j\omega \left\{ \int_\Omega J^{imp} \cdot \bar{F} dx + \int_{\Gamma_2} J_S^{imp} \cdot \bar{F} dS \right\} \quad \forall F \in H_D(\mathrm{curl};\Omega) \\ \displaystyle - \int_\Omega (\omega^2 \epsilon - j\omega\sigma) E \cdot \nabla \bar{q} \, dx = -j\omega \left\{ \int_\Omega J^{imp} \cdot \nabla \bar{q} \, dx + \int_{\Gamma_2} J_S^{imp} \cdot \nabla \bar{q} dS \right\} \quad \forall q \in H_D^1(\Omega) \,. \end{array} \right.$$

MAXWELL'S EQUATIONS and hp-FEM

De Rham diagram

De Rham diagram is critical to the theory of FE discretizations of Maxwell's equations.

$$
\begin{array}{ccccccccccc}
\mathbb{R} & \longrightarrow & W & \xrightarrow{\nabla} & Q & \xrightarrow{\nabla\times} & V & \xrightarrow{\nabla\circ} & L^2 & \longrightarrow & 0 \\
\Big\downarrow id & & \Big\downarrow \Pi & & \Big\downarrow \Pi^{\mathrm{curl}} & & \Big\downarrow \Pi^{\mathrm{div}} & & \Big\downarrow P & & \\
\mathbb{R} & \longrightarrow & W^{\mathrm{p}} & \xrightarrow{\nabla} & \mathrm{Q}^{\mathrm{p}} & \xrightarrow{\nabla\times} & \mathrm{V}^{\mathrm{p}} & \xrightarrow{\nabla\circ} & W^{\mathrm{p}-1} & \longrightarrow & 0\,.
\end{array}
$$

This diagram relates two exact sequences of spaces, on both continuous and discrete levels, and corresponding interpolation operators.

hp-FINITE ELEMENTS

Exponential convergence rates

for a number of regular and SINGULAR problems

for optimal hp-grids

in the asymptotic range (theoretical and numerical results), and

in the pre-asymptotic range (numerical results).

Smaller dispersion (pollution) error

as *p* increases.

More geometrical details captured

as *h* decreases.

hp-FINITE ELEMENTS

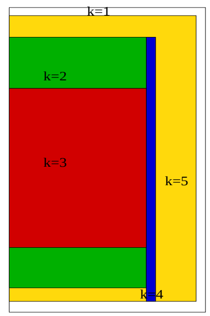
2Dhp90, 3Dhp90: main features

- **Isoparametric triangles, squares and hexahedras.**
- H^1 **and** $H(curl)$ **dofs.**
- **Isotropic and anisotropic mesh refinements.**
- **Geometrical Modeling Package (GMP).**
- **New data structure in Fortran 90.**
- **Constrained information reconstructed (not stored).**
- **Two levels of logical operations:**
  1. **operations for nodes - problem independent.**
  2. **operations for nodal dof - problem dependent.**
- **Fully automatic** hp**-adaptive strategy.**

—provides exponential convergence rates—

Numerical Results

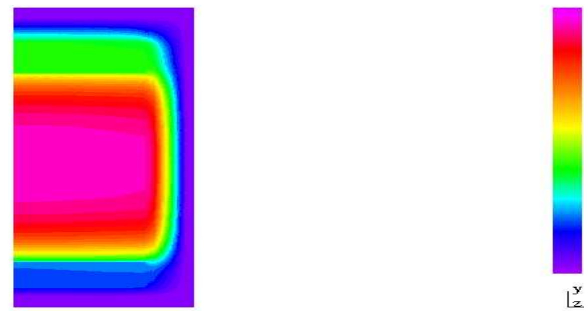
Orthotropic heat conduction example



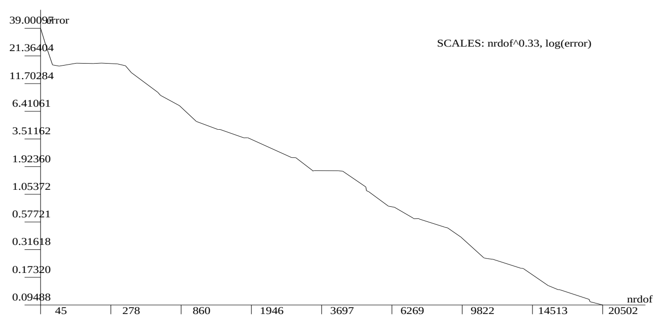
Equation: $\nabla(\mathrm{K}\nabla u) = f^{(k)}$

$$\mathrm{K} = \mathrm{K}^{(k)} = \begin{bmatrix} K_x^{(k)} & 0 \\ 0 & K_y^{(k)} \end{bmatrix}$$

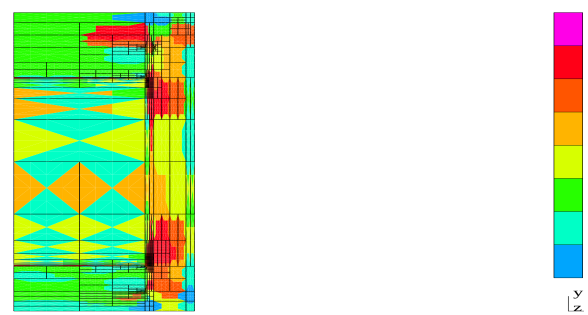
$$K_x^{(k)} = (25,\ 7,\ 5,\ 0.2,\ 0.05)$$
$$K_y^{(k)} = (25,\ 0.8,\ 0.0001,\ 0.2,\ 0.05)$$



Solution: unknown
Boundary Conditions:
$K^{(i)}\nabla u \cdot n = g^{(i)} - \alpha^{(i)} u$



Convergence history
(tolerance error = 0.1 %)

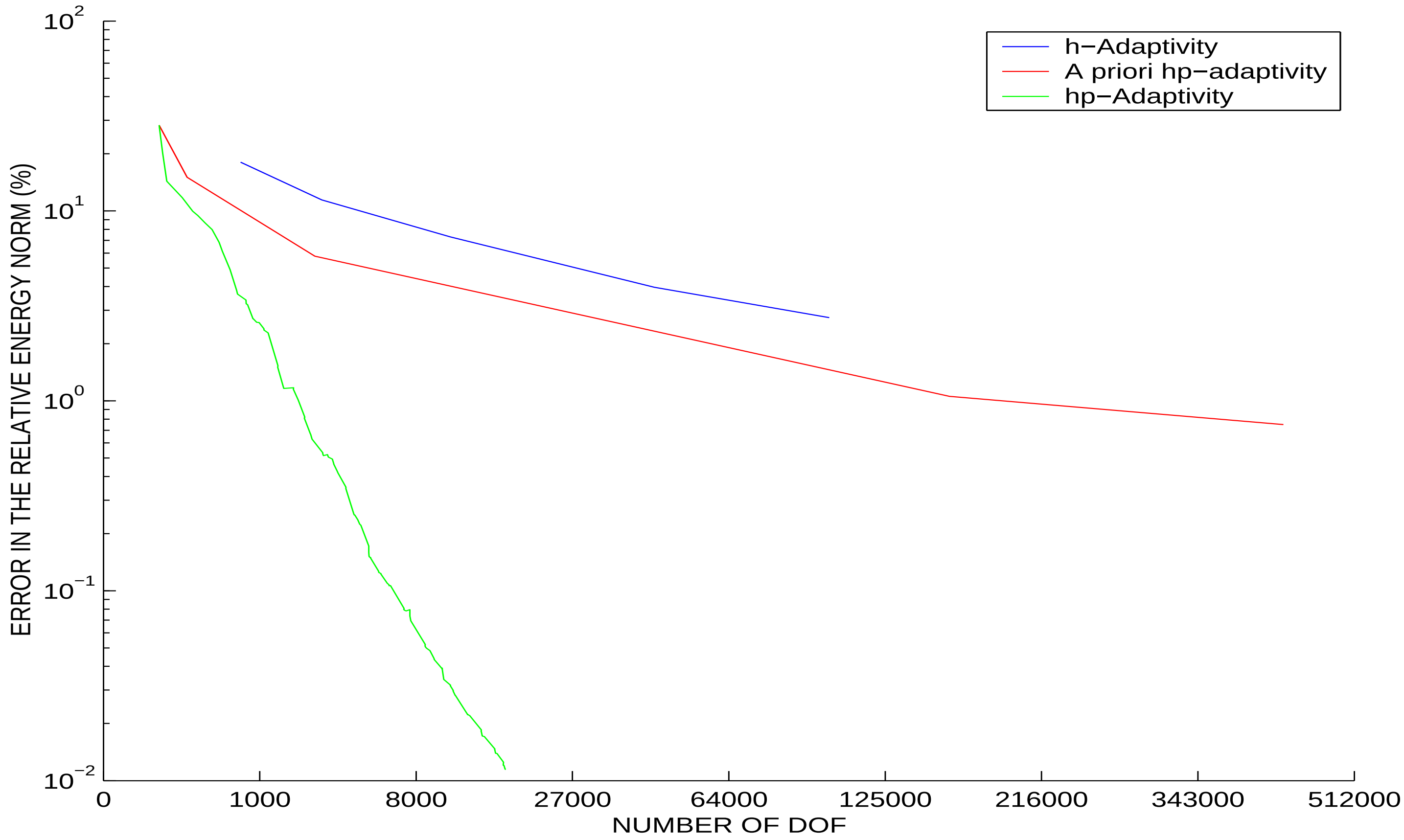


Final hp grid

hp-FINITE ELEMENTS

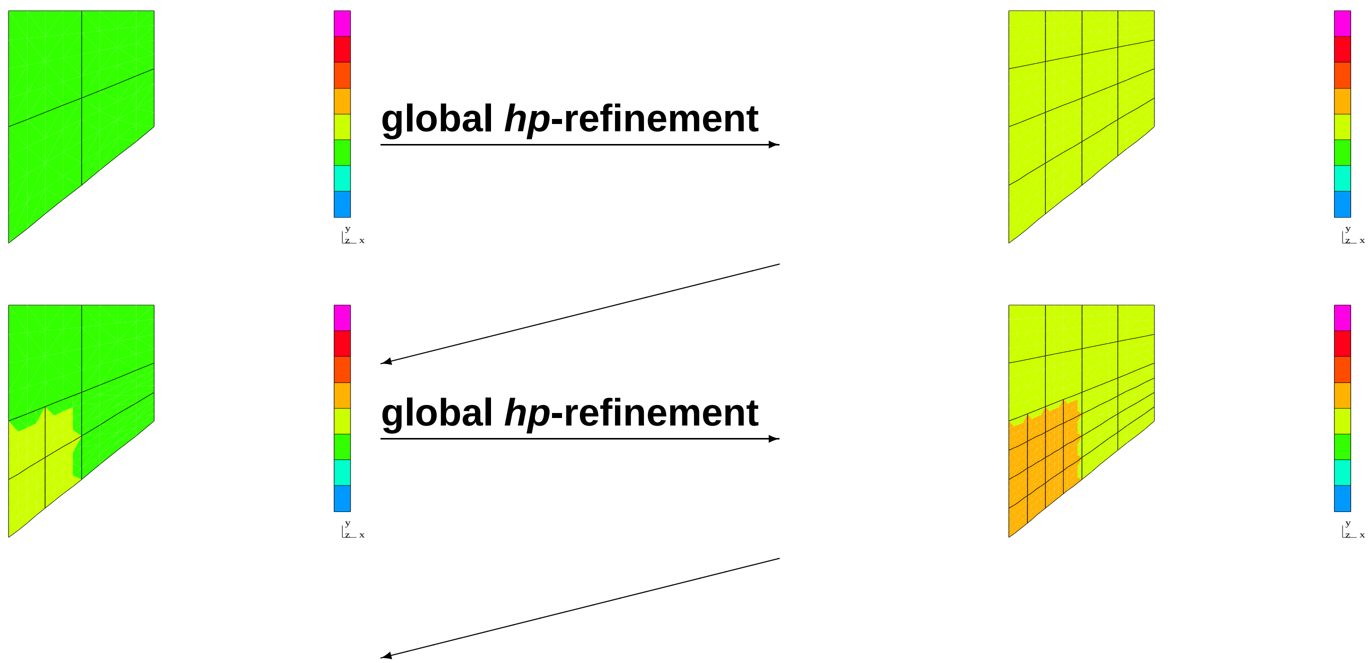
Convergence comparison

Orthotropic heat conduction example



hp-FINITE ELEMENTS

Fully automatic *hp*-adaptive strategy



A Two Grid Solver for SPD problems

We seek x such that $Ax = b$. Consider the following iterative scheme:

$$r^{(n+1)} = [I - \alpha^{(n)} AS] r^{(n)}$$
$$x^{(n+1)} = [I - \alpha^{(n)} S] r^{(n)}$$

where S is a matrix, and $\alpha^{(n)}$ is a relaxation parameter. $\alpha^{(n)}$ *optimal* if:

$$\alpha^{(n)} = \arg \ \min \parallel x^{(n+1)} - x \parallel_A = \frac{(A^{-1} r^{(n)}, S r^{(n)})_A}{(S r^{(n)}, S r^{(n)})_A}$$

Then, we define our two grid solver as:

1 Iteration with $S = S_F = \sum A_i^{-1} \quad +$

1 Iteration with $S = S_C = P A^{-1} R$

A Two Grid Solver for Electromagnetics

We seek x such that $Ax = b$. Consider the following iterative scheme:

$$r^{(n+1)} = [I - \alpha^{(n)} AS] r^{(n)}$$
$$x^{(n+1)} = [I - \alpha^{(n)} S] r^{(n)}$$

where S is a matrix, and $\alpha^{(n)}$ is a relaxation parameter. $\alpha^{(n)}$ *optimal* if:

$$\alpha^{(n)} = \arg \ \min \| x^{(n+1)} - x \|_A = \frac{(A^{-1} r^{(n)}, S r^{(n)})_A}{(S r^{(n)}, S r^{(n)})_A}$$ **(NOT COMPUTABLE)**

Then, we define our two grid solver for Electromagnetics as:

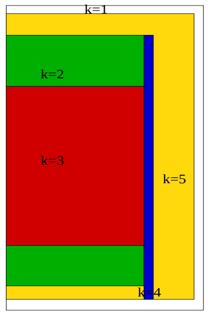
1 Iteration with $S = S_F = \sum A_i^{-1} \quad +$

1 Iteration with $S = S_{grad} = \sum B_i^{-1} \quad +$

1 Iteration with $S = S_C = P A^{-1} R$

Numerical Results

Orthotropic heat conduction example



Equation: $\nabla(\mathrm{K}\nabla u) = f^{(k)}$

$$\mathrm{K} = \mathrm{K}^{(k)} = \begin{bmatrix} K_x^{(k)} & 0 \\ 0 & K_y^{(k)} \end{bmatrix}$$

$$K_x^{(k)} = (25,\ 7,\ 5,\ 0.2,\ 0.05)$$
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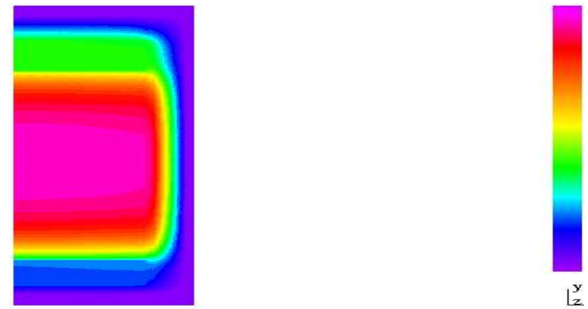
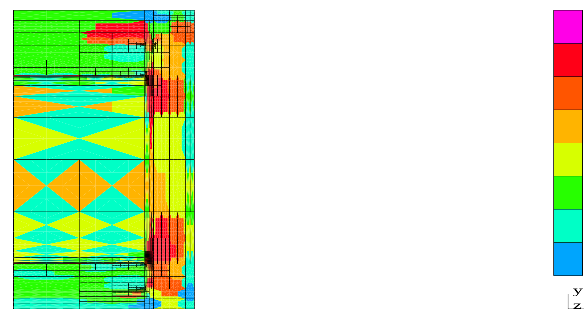


Solution: unknown
Boundary Conditions:
$$K^{(i)}\nabla u \cdot n = g^{(i)} - \alpha^{(i)} u$$

Convergence history
(tolerance error = 0.1 %)

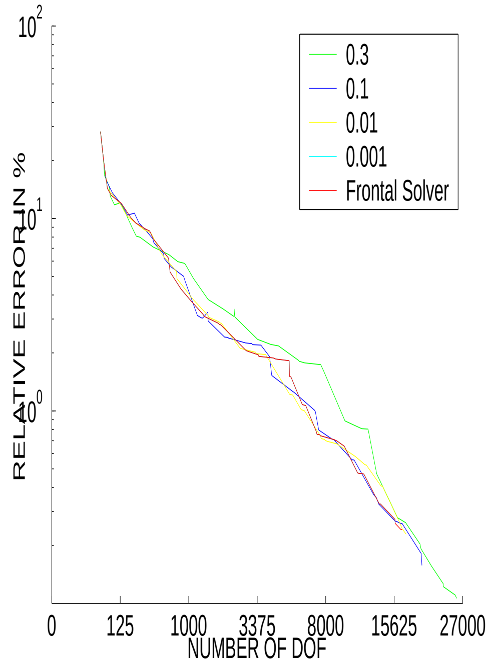


Final hp grid

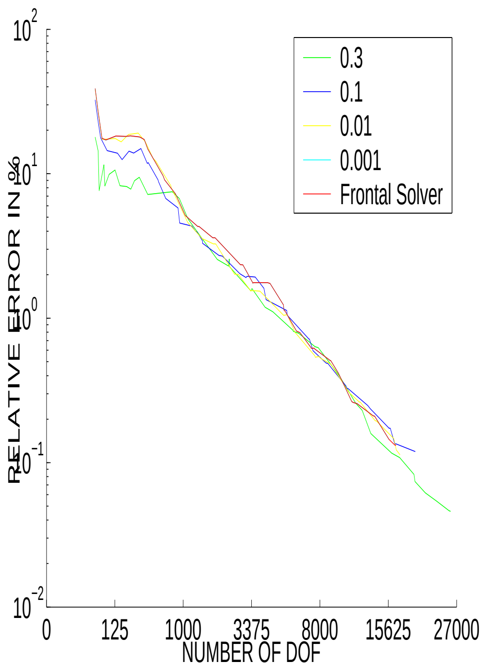
Numerical Results

Guiding automatic hp-refinements

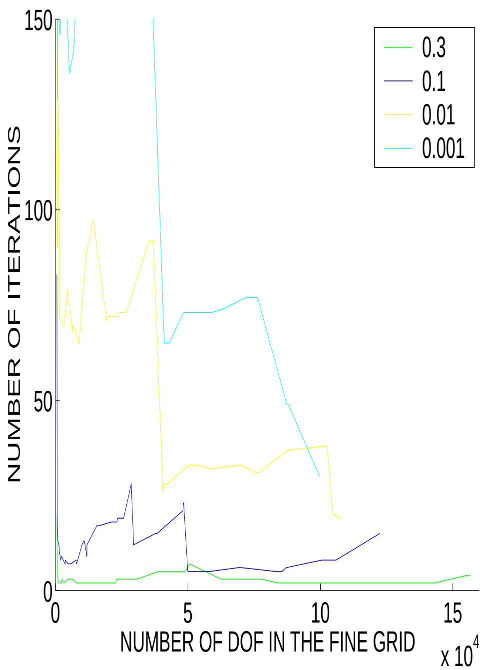
Orthotropic heat conduction. Guiding hp-refinements with a partially converged solution.



Energy error estimate



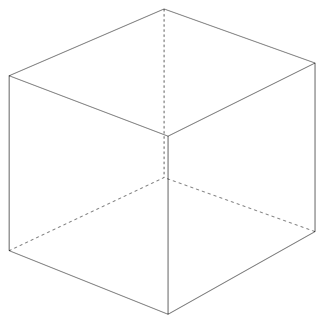
Discretization error estimate



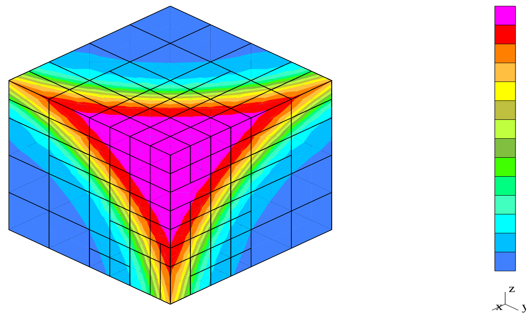
Number of iterations

Numerical Results

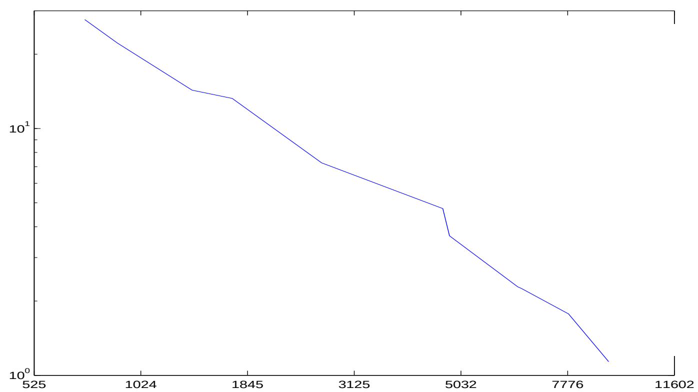
3D shock like solution example



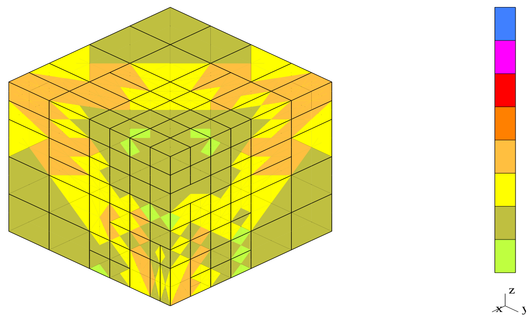
Equation: $-\Delta u = f$
Geometry: unit cube



Solution: $u = atan(20 * \sqrt{r} - \sqrt{3}))$
$r = (x - .25) * *2 + (y - .25) * *2 + (z - .25) * *2$
Dirichlet Boundary Conditions



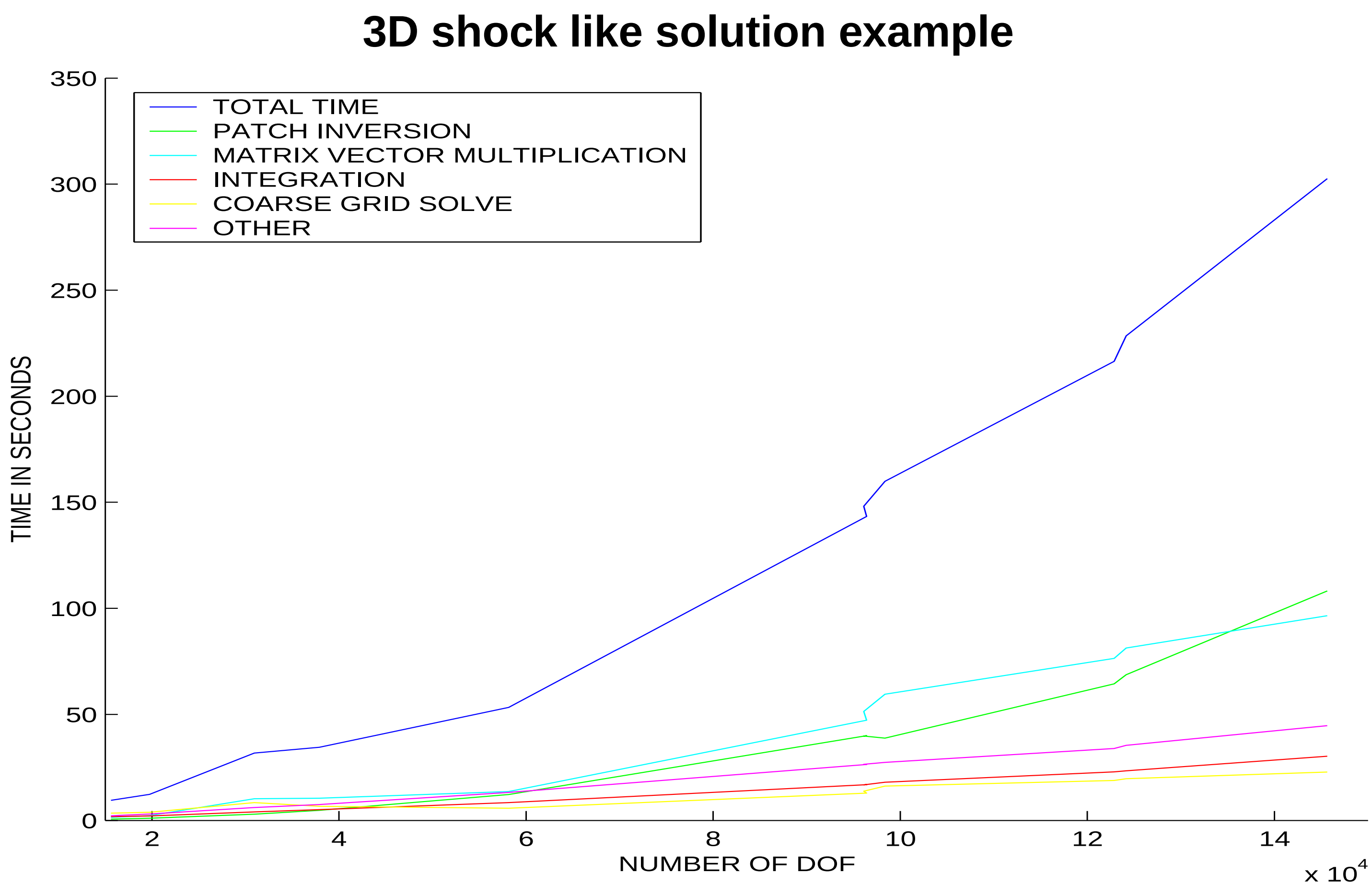
Convergence history
(tolerance error = 1%)



Final hp grid

Numerical Results

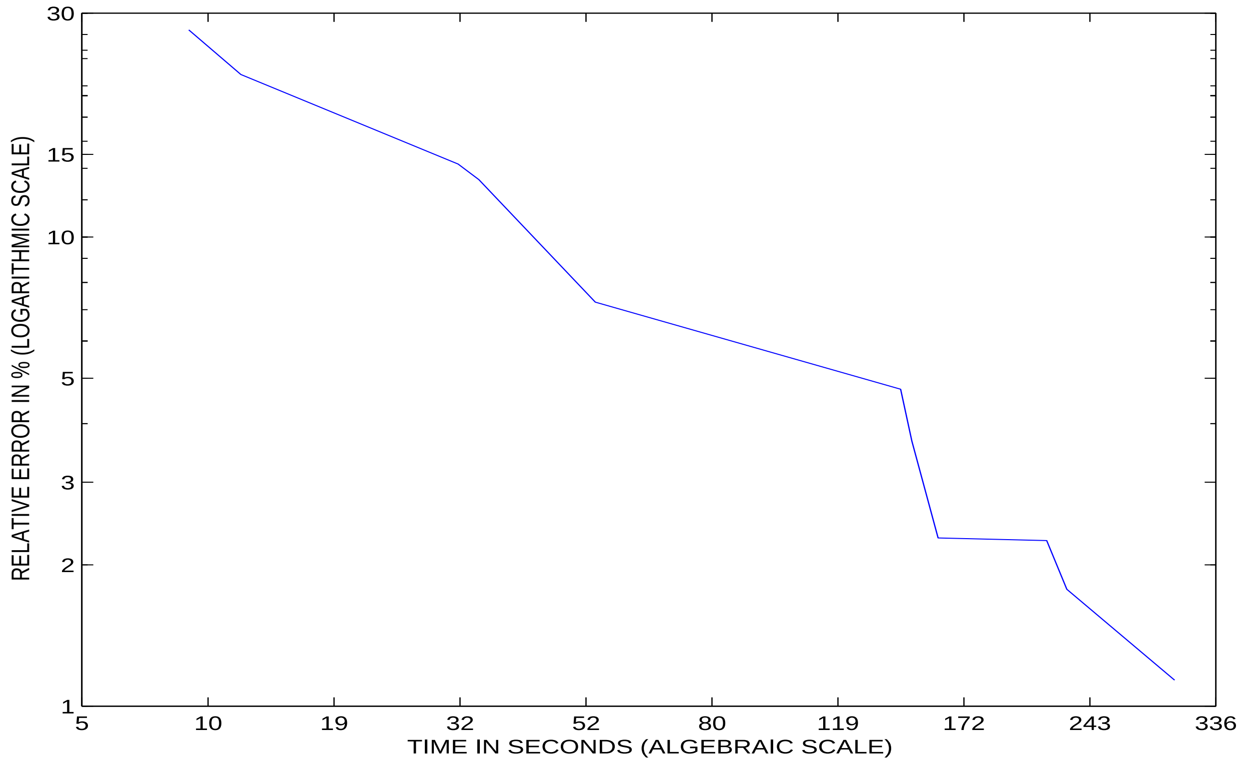
Performance of the two grid solver



Numerical Results

Convergence history

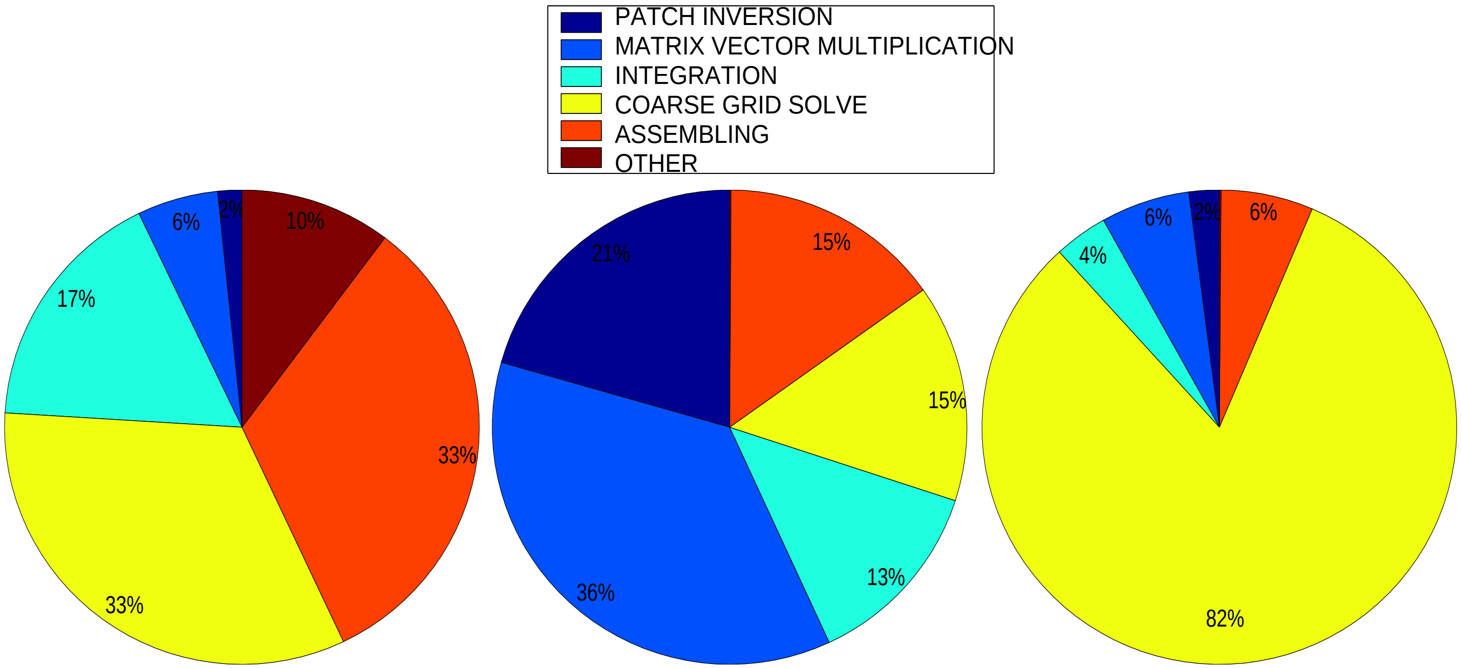
3D shock like solution example.
Scales: ERROR VS TIME.



Numerical Results

Performance of the two grid solver

3D shock like solution problem

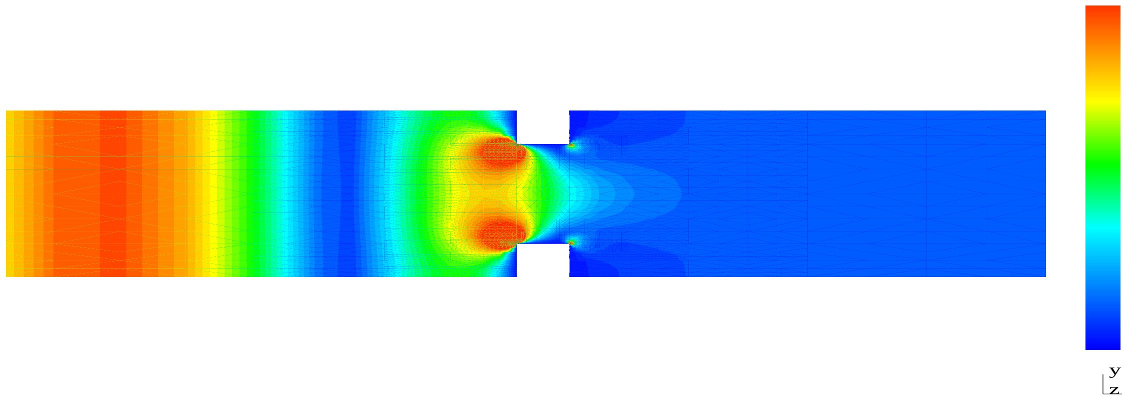


Nrdofs ≈ 2.15 Million	Nrdofs ≈ 0.27 Million	Nrdofs ≈ 2.15 Million
Total time ≈ 8 minutes	Total time ≈ 10 minutes	Total time ≈ 50 minutes
Memory* ≈ 1.0 Gb	Memory* ≈ 2.0 Gb	Memory* ≈ 3.5 Gb
p=2	p=8	p=4

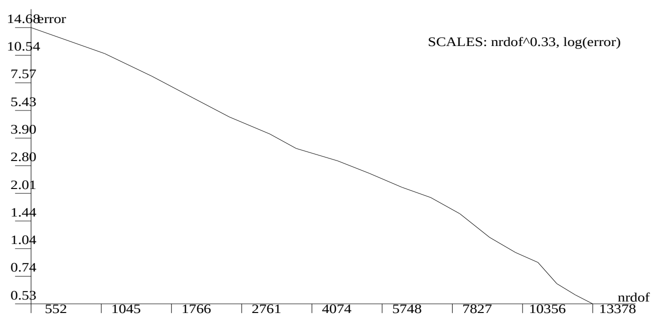
***Memory = memory used by nonzero entries of stiffness matrix**
In core computations, IBM Power4 1.3 Ghz processor.

Numerical Results

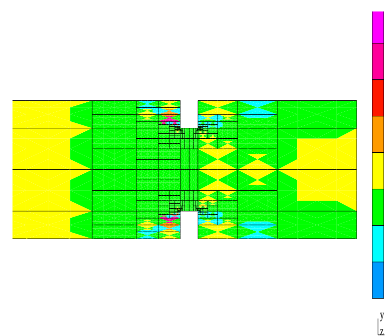
Waveguide example



Module of Second Component of Magnetic Field



Convergence history
(tolerance error = 0.5 %)

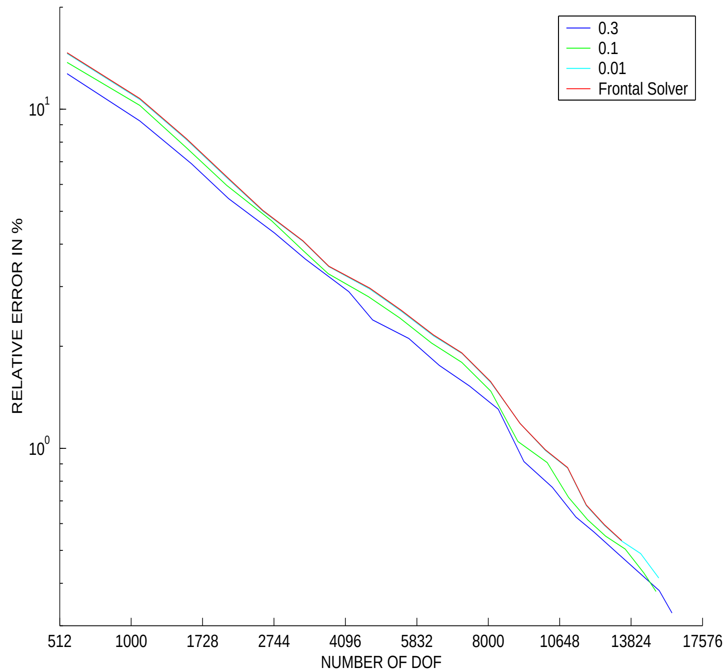


Final hp-grid

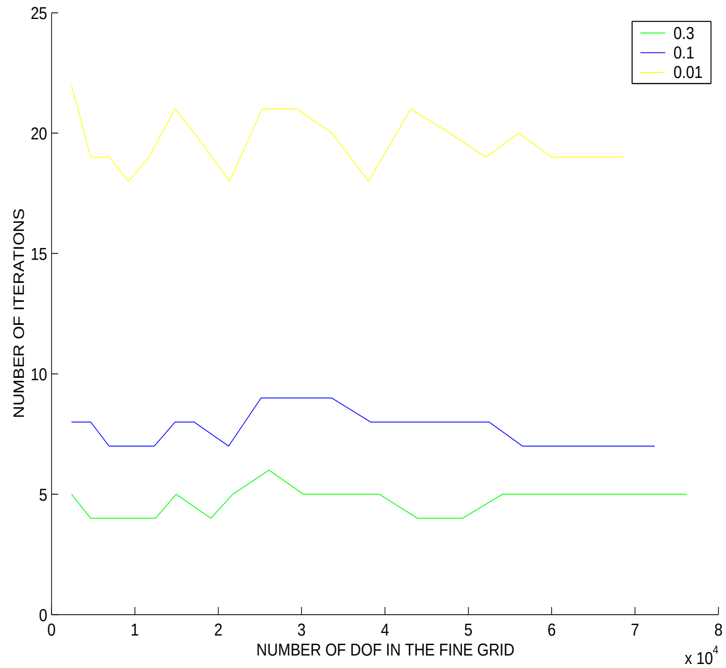
Numerical Results

Guiding automatic hp-refinements

Waveguide example. Guiding hp-refinements with a partially converged solution.



Discretization error estimate



Number of iterations

Conclusions and Future Work

AN EFFICIENT FULLY AUTOMATIC HP-ADAPTIVE FINITE ELEMENT ADAPTIVE PACKAGE FOR ELECTROMAGNETICS IS POSSIBLE.

Future Work:

Parallelize the code.

Goal-oriented adaptivity.

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