SOME REMARKS ABOUT THE UNCERTAINTY PRINCIPLE

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Rubio de Francia Lecture

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Summary

- Uncertainty Principle (Heisenberg).
- Examples: Schrödinger stationary equation (Heisenberg, Coulomb, Hardy).
- Dirac Equation (Pauli, Dirac, Hardy).
- Fourier Transform (Heisenberg, Paley–Wiener, Hardy, Morgan).
- A "dynamic" uncertainty principle: Schrödinger evolution equation and wave packets.

Uncertainty Principle

$$S \text{ symmetric} \qquad \langle Sf, f \rangle = \langle f, Sf \rangle$$

$$\mathcal{A} \text{ skewsymmetric} \qquad \langle \mathcal{A}f, f \rangle = -\langle f, \mathcal{A}f \rangle$$

$$\big\langle (S+\mathcal{A})f, (S+\mathcal{A})f \big\rangle = \langle Sf, Sf \rangle + \langle \mathcal{A}f, \mathcal{A}f \rangle + \big\langle (S\mathcal{A} - \mathcal{A}S)f, f \big\rangle$$

Hence

$$\big\langle (\mathcal{A}S - S\mathcal{A})f, f \big\rangle \leq \|Sf\|_{L^2}^2 + \|\mathcal{A}f\|_{L^2}^2$$

$$\mathcal{A} \hookrightarrow \pm\lambda\mathcal{A} \qquad ; \qquad S \hookrightarrow \frac{1}{\lambda} S$$

$$\Big|\big\langle(\mathcal{A}S - S\mathcal{A})f, f\big\rangle\Big| \leq 2\|Sf\|_{L^2}\|\mathcal{A}f\|_{L^2}$$

$$\mathcal{A} \hookrightarrow \mathcal{A} - \big\langle\mathcal{A}f, f\big\rangle \mathbb{1} \quad ; \quad S \hookrightarrow S - \langle Sf, f\rangle \mathbb{1}$$

Examples

1.–) $$Sf = xf \qquad ; \qquad \mathcal{A}f = f'$$

$$\mathcal{A}S - S\mathcal{A} = \frac{d}{dx}x - x\frac{d}{dx} = \mathbb{1}.$$

$$\|f\|_{L^2}^2 \leq 2\|f'\|_{L^2}\|xf\|_{L^2}.$$

$$(S + \mathcal{A})f = 0 \iff f(x) = ce^{-x^2/2}$$

$$(S - \mathcal{A})(S + \mathcal{A})f = -f'' + x^2 f - f = 0 \qquad \text{Harmonic Oscillator}$$

2.–)

$$Sf = \frac{x}{|x|} \qquad ; \qquad \mathcal{A}f = \nabla f$$

$$\mathcal{A}S - S\mathcal{A} = \frac{d-1}{|x|} \qquad (d \geq 2)$$

$$\int \frac{d-1}{|x|}|f|^2 \leq 2\left(\int |f|^2\right)^{1/2}\left(\int |\nabla f|^2\right)^{1/2}$$

$$(S+\mathcal{A})f = 0 \quad \Longleftrightarrow \quad f(x) = ce^{-|x|}$$

$$(S-\mathcal{A})(S+\mathcal{A})f = -\Delta f - \frac{d-1}{|x|}f + f = 0 \qquad \text{Coulomb Potential}$$

3.–) $Sf = \frac{d-2}{2}\frac{x}{|x|^2}f \quad ; \quad \mathcal{A}f = \nabla f \qquad d \geq 3$

$$(\mathcal{A}S - S\mathcal{A})f = \frac{(d-2)^2}{2}\frac{1}{|x|^2}f.$$

$$\left(\int \frac{(d-2)^2}{2|x|^2}|f|^2\right)^{1/2} \leq \left(2\int |\nabla f|^2\right)^{1/2}$$ Hardy's Inequality

$$(S+\mathcal{A})f = 0 \iff f = |x|^{1-\frac{d}{2}} \qquad \left(\nabla f \notin L^2\right)$$

$$(S-\mathcal{A})(S+\mathcal{A})f = -\Delta f - \frac{(d-2)^2}{4|x|^2}f = 0$$

$$\left(-\Delta - \frac{(d-2)^2}{4|x|^2} \text{ is not self–adjoint}\right)$$

Dirac Equation

Angular momentum:

$$L = x \wedge i\nabla\,; \quad L = (L_1, L_2, L_3)\,, \quad x \in \mathbb{R}^3.$$

CR: $[L_1; L_2] = -iL_3\,; \quad |\langle L_3 f, f\rangle| \leq 2\,\|L_1 f\|\,\|L_2 f\|\,.$

Pauli matrices:

$$\sigma = (\sigma_1, \sigma_2, \sigma_3)\,; \quad \sigma_j\sigma_k + \sigma_k\sigma_j = 2\delta_{jk}\,, \quad \overline{\sigma}^t = \sigma.$$

CR: $[\sigma_1; \sigma_2] = -2i\sigma_3\,; \quad |\langle \sigma_3 f, f\rangle| \leq \|\sigma_1 f\|\,\|\sigma_2 f\|\,.$

$$\langle(\sigma_1 L_1 + \sigma_2 L_2)\,f, (\sigma_1 L_1 + \sigma_2 L_2)\,f\rangle$$
$$\leq \|\sigma_1 L_1 f\|^2 + \|\sigma_2 L_2 f\|^2 + \langle(\sigma_2 L_2 \sigma_1 L_1 + \sigma_1 L_1 \sigma_2 L_2)\,f, f\rangle$$

$$\langle \sigma_3 L_3 f, f\rangle \leq \|\sigma_1 L_1 f\|^2 + \|\sigma_2 L_2 f\|^2$$

As a consequence:

$$\|(1 + \sigma \cdot L)f\| \geq \|f\|.$$

(with J. Dolbeault, M. Esteban, M. Loss)

$$\int |\psi|^2 \frac{dx}{|x|} \leq \int |x| \, |\sigma \cdot \nabla \psi|^2 \, dx$$

$$\int |\psi|^2 \frac{dx}{|x|} \leq \int_{\mathbb{R}^3} \left(\frac{|\sigma \cdot \nabla \psi|^2}{1 + \frac{1}{|x|}} + |\psi|^2 \right) dx$$ (M. Esteban, E. Sere)

Ground State Hydrogen Atom

$$\mathcal{H} = \alpha \cdot i\nabla + m\beta + \frac{v}{|x|}$$

$$\alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix} \qquad j \in \{1, 2, 3\} \qquad ; \qquad \beta = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$$

$$m = 1 \qquad \lambda_1^v = \sqrt{1 - v^2} \qquad v \in (0, 1)$$

(with N. Arrizabalaga and J. Duoandikoetxea)

Theorem 1.– $\mathcal{H} = \alpha \cdot i\nabla + V\mathbb{1}$ with $|V(x)| \leq \dfrac{\upsilon}{|x|}$, $\upsilon < 1$ and $D(\mathcal{H}) = \left\{\psi \in L^2 \text{ s.t. } \mathcal{H}\psi \in L^2\right\}$ is self–adjoint.

- Rellich, Schmincke: If $\upsilon > \dfrac{\sqrt{3}}{2}$ then $\mathcal{H}$ is not essentially self–adjoint in $\mathcal{C}_0^\infty\left(\mathbb{R}^3 \setminus \{0\}\right)$.

- Schmincke, Wust, Nenciu: $\exists!$ self–adjoint extension such that $\displaystyle\int |\psi|^2 \frac{dx}{|x|} < +\infty$.

$$\int |\psi|^2 \frac{dx}{|x|} \leq \int |(\alpha \cdot i\nabla + i\mathbb{1})\,\psi|^2\,|x|dx$$

The constant is sharp.

Uncertainty Principle (Fourier Transform)

Define for $\xi \in \mathbb{R}^n$

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)dx$$

- $\widehat{\nabla f}(\xi) = -i\xi \widehat{f}(\xi)$
- $\|f\|_{L^2} = \|\widehat{f}\|_{L^2}$
- Heisenberg
$$\|f\|_{L^2}^2 \leq 2\|xf\|_{L^2}\|\xi\widehat{f}\|_{L^2}.$$
Identity holds iff $f = ce^{-x^2/2}$.

Hardy's uncertainty principle (Cowling, Price)

Define for $\xi \in \mathbb{R}^n$

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\, dx$$

Assume that

$$\begin{aligned} f(x)e^{\frac{|x|^2}{\alpha^2}} &\in L^2 \\ \widehat{f}(\xi)e^{\frac{|\xi|^2}{\beta^2}} &\in L^2 \\ \alpha^2\beta^2 &\leq 4. \end{aligned}$$

Then

$$f \equiv 0$$

Morgan and Paley-Wiener

If

$$\int |f|^2 e^{-|x|^p} dx < +\infty$$

$$\int |\widehat{f}|^2 e^{-|\xi|^{p'}} d\xi < +\infty$$

$$1 < p < \infty \qquad \frac{1}{p} + \frac{1}{p'} = 1$$

then $u \equiv 0$.

Paley–Wiener

If $\int |f(x)| e^{-\epsilon |x|} dx < +\infty, \ \epsilon > 0,$

then $\widehat{f}$ can not have compact support.

About the proofs in the constant coefficient case (key words):

- Uncertainty principle:
  - Positive Commutators.
- Hardy's theorem:
  - Analyticity, Cauchy–Riemann equations, Log convexity.
  - Liouville's theorem.

Log-Convexity (a dynamic uncertainty principle)

S an $\mathcal{A}$ as before

$$\partial_t v = (S + \mathcal{A})v$$

$$H(t) = \langle v, v\rangle$$

$$\begin{aligned}\dot{H}(t) &= \langle v_t, v\rangle + \langle v, v_t\rangle \\ &= \big\langle (S+\mathcal{A})v, v\big\rangle + \big\langle v, (\mathcal{A}+S)v\big\rangle \\ &= 2\langle Sv, v\rangle.\end{aligned}$$

$$\begin{aligned}\ddot{H}(t) &= 2\big\langle Sv_t, v\big\rangle + 2\big\langle Sv, v_t\big\rangle \\ &= 2\big\langle (S+\mathcal{A})v, Sv\big\rangle + 2\big\langle Sv, (S+\mathcal{A})v\big\rangle \\ &= 4\langle Sv, Sv\rangle + 2\big\langle (S\mathcal{A} - \mathcal{A}S)v, v\big\rangle\end{aligned}$$

$$
\begin{aligned}
(\lg H(t))^{\cdot\cdot} &= \left(\frac{\dot{H}}{H}\right)^{\cdot} = \frac{\ddot{H}H - \dot{H}^2}{H^2} \\
&= \frac{1}{\langle v, v\rangle}\left\{4\langle Sv, Sv\rangle\langle v, v\rangle - 4\langle Sv, v\rangle^2 + 2\Big\langle(S\mathcal{A} - \mathcal{A}S)v, v\Big\rangle\right\} \\
&\geq 2\Big\langle(S\mathcal{A} - \mathcal{A}S)v, v\Big\rangle
\end{aligned}
$$

Hence if $2\Big\langle(S\mathcal{A} - \mathcal{A}S)v, v\Big\rangle \geq 0$ then

$$H(t) \leq H(0)^{1-t} H(1)^t$$

More generally if

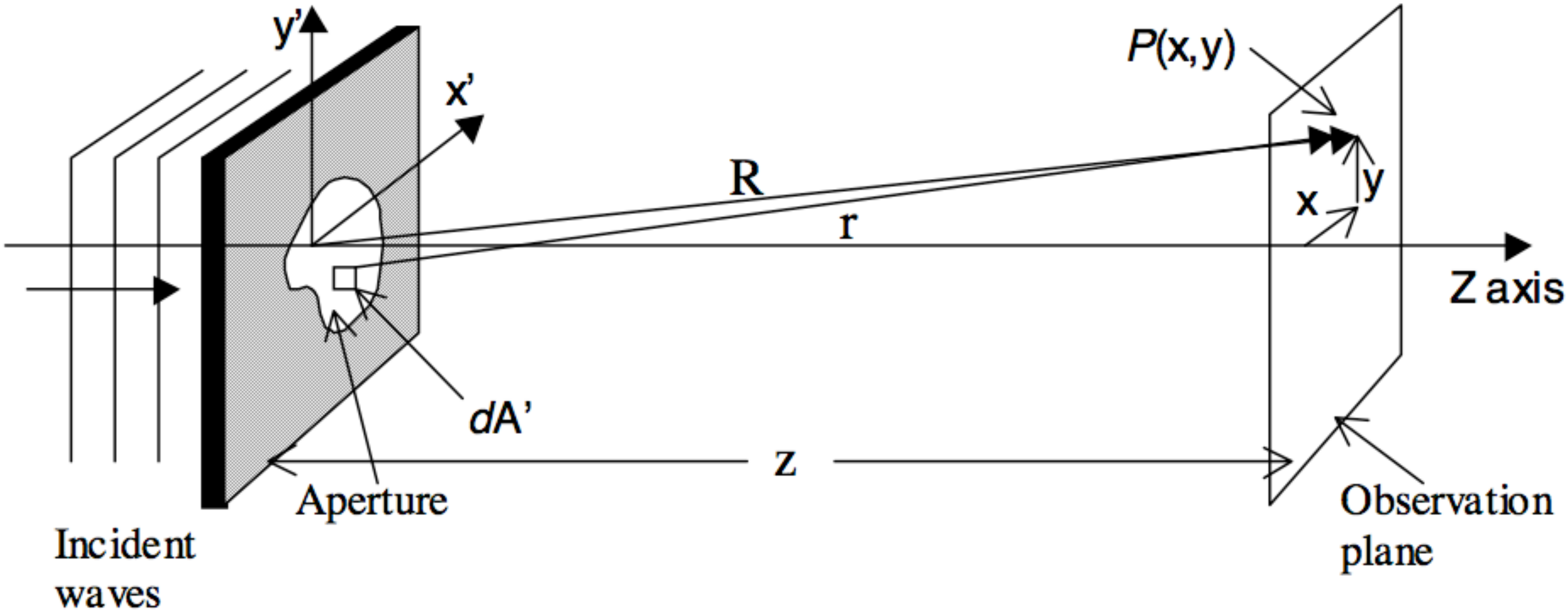
$$2\Big\langle(S\mathcal{A} - \mathcal{A}S)v, v\Big\rangle \geq \psi(t)\langle v, v\rangle,$$

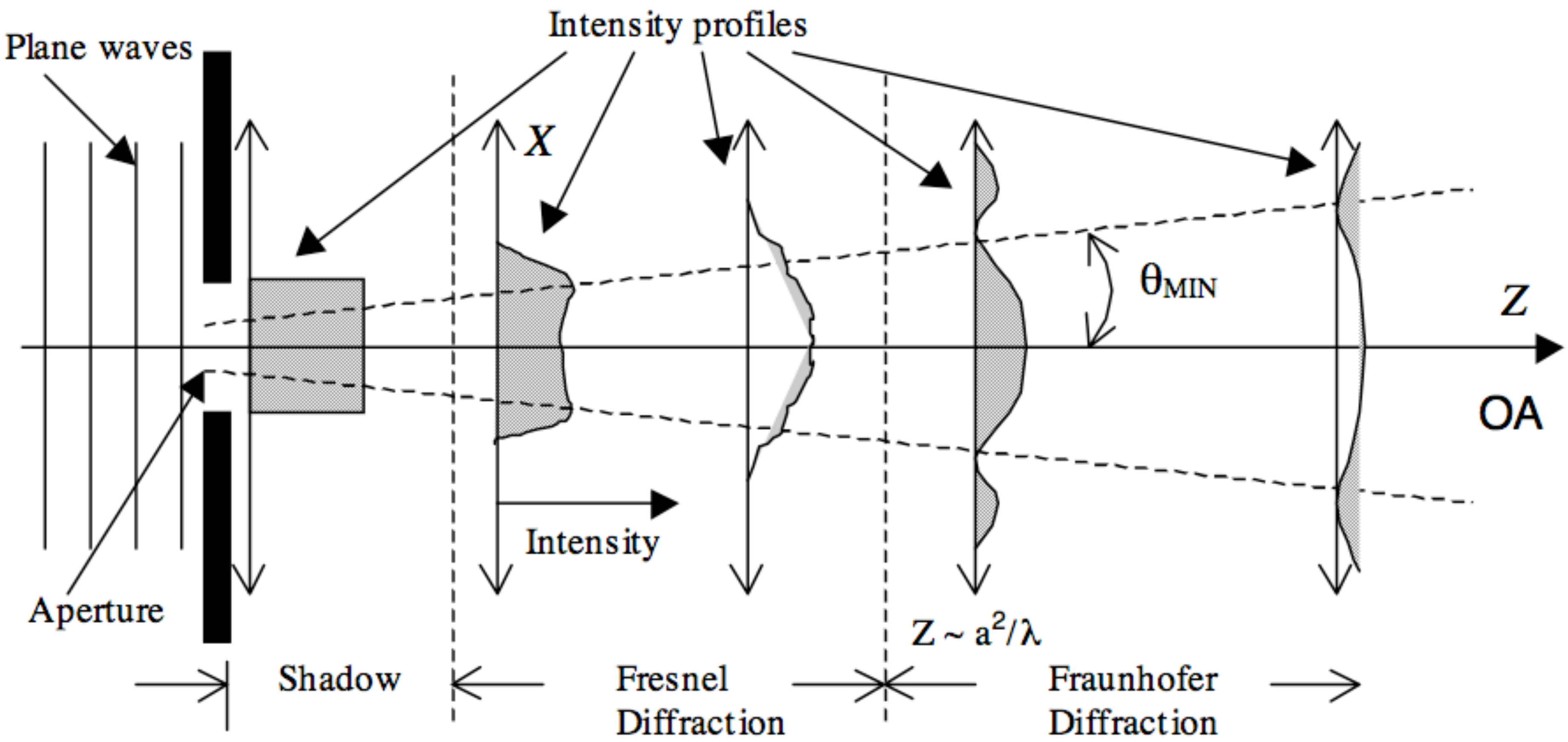
then

$$H(t)e^{-B(t)} \leq H(0)^{1-t} H(1)^t,$$

with

$$\ddot{B} = \Psi(t), \quad B(0) = B(1) = 0.$$


y'
x'
P(x,y)
R
r
y
x
Z axis
dA'
z
Aperture
Incident
waves
Observation
plane



Intensity profiles
Plane waves
X
θMIN
Z
OA
Intensity
Aperture
Z ~ a2/λ
Shadow
Fresnel
Diffraction
Fraunhofer
Diffraction

Free Schrödinger Equation

S.E. $\begin{cases} \partial_t u = i\Delta u & x \in \mathbb{R}^n \quad t \in \mathbb{R} \\ u(0) = u_0 \end{cases}$

$$u(x,t) = \frac{1}{(it)^{n/2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy$$

$$= \frac{1}{(it)^{n/2}} e^{i\frac{|x|^2}{4t}} \int_{\mathbb{R}^n} e^{-i\frac{x}{2t}\cdot y} e^{i\frac{|y|^2}{4t}} u_0(y)dy$$

$$f(y) = e^{i\frac{|y|^2}{4t}} u_0(y),$$

$$\widehat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} f(y)dy.$$

Hence

$$u(0)e^{\frac{|y|^2}{\alpha^2}} \in L^2 \quad \Longleftrightarrow \quad f e^{\frac{|y|^2}{\alpha^2}} \in L^2$$

$$u(T)e^{\frac{|x|^2}{\beta^2}} \in L^2 \quad \Longleftrightarrow \quad \widehat{f}\left(\frac{x}{2T}\right) e^{\frac{|x|^2}{\beta^2}} \in L^2.$$

Hardy's uncertainty principle:

$$\alpha\beta \leq 4T \quad \text{then} \quad u \equiv 0$$

Time/Energy uncertainty principle (Wave Packets)

$$u(x,t) = \frac{1}{(t+i)^{n/2}} e^{-\frac{|x|^2}{4(t+i)}}$$

$$|u|^2 = \frac{1}{(t^2+1)^{n/2}} e^{-\frac{|x|^2}{2(t^2+1)}}$$

- $u_R(x,t) = u\big(Rx, R^2 t\big)$ is also a solution.
- $u^{t_0}(x,t) = u\big(x, t - t_0\big)$ is also a solution.

Schrödinger Equation with a Potential

$$u = u(x,t)$$

$$\partial_t u = i(\Delta + V)u \qquad x \in \mathbb{R}^n$$

$$\left.\begin{array}{l} \bullet \quad e^{\frac{|x|^2}{\alpha^2}} u(0) \in L^2 \\ \bullet \quad e^{\frac{|x|^2}{\beta^2}} u(T) \in L^2 \\ \bullet \quad \alpha\beta \ \text{ small enough} \end{array}\right\} \implies \quad u \equiv 0$$

With C. E. Kenig, L. Escauriaza and G. Ponce

Theorem 2.– $u \in \mathcal{C}\left([0,T] : L^2(\mathbb{R}^n)\right)$ solution of

$$\partial_t u = i(\Delta + V)u \qquad x \in \mathbb{R}^n \quad t \in [0,T].$$

$$u(0)e^{\frac{|x|^2}{\alpha^2}} \in L^2 \qquad ; \qquad u(T)e^{\frac{|x|^2}{\beta^2}} \in L^2,$$

and $\alpha\beta < 4T$, then $u \equiv 0$.

Hypothesis on the potential:

H1 $V = V_0(x) +$ perturbation with a gaussian decay
V_0 real and bounded

H1* $V = V(x,t)$

$$\lim_{R\to\infty} \int_0^T \sup_{|x|\geq R} |V| dt = 0$$

Theorem 3.–There exists a non-trivial $u \in \mathcal{C}\left(\mathbb{R} : L^2(\mathbb{R}^n)\right)$ solution of

$$\partial_t u = i(\Delta + V)u \qquad x \in \mathbb{R}^n \quad t \in \mathbb{R},$$

with $V = V(x,t) \in \mathbb{C}$ and $|V(x,t)| \leq \frac{C}{1+|x|^2}$, such that

$$u(0)e^{\frac{|x|^2}{\alpha^2}} \in L^2 \qquad ; \qquad u(1)e^{\frac{|x|^2}{\beta^2}} \in L^2,$$

and $\alpha^2\beta^2 = 4$.

A Particular Example

- $\partial_t u = i\Delta u$
- $e^{\frac{|x|^2}{2}} u = v$
- $H(t) = \|v(t)\|^2_{L^2} = \langle v(t), v(t) \rangle$
- $$\begin{aligned} \partial_t v &= \left(e^{\frac{|x|^2}{2}} i\Delta e^{-\frac{|x|^2}{2}} \right) v \\ &= i \sum_j e^{\frac{|x|^2}{2}} \partial_j^2 e^{-\frac{|x|^2}{2}} v \\ &= i \sum_j e^{\frac{|x|^2}{2}} \partial_j e^{-\frac{|x|^2}{2}} e^{\frac{|x|^2}{2}} \partial_j e^{-\frac{|x|^2}{2}} v \\ &= i \sum_j (x_j - \partial_j)(x_j - \partial_j) v \\ &= i \left(|x|^2 - 2x \cdot \nabla - 1 + \Delta \right) v \qquad \textbf{(C–R)} \end{aligned}$$

$$\partial_t v = (S + \mathcal{A})v \quad ; \quad Sv = -i(2x \cdot \nabla + 1)v \quad ; \quad \mathcal{A}v = i(\Delta + |x|^2)$$

$$S\mathcal{A} - \mathcal{A}S = -4\Delta + 4|x|^2$$

$$\big\langle S\mathcal{A} - \mathcal{A}Sv, v\big\rangle = 4(\|\nabla v\|^2 + \|xv\|^2) \geq 4\langle v, v\rangle.$$

Hence

$$(\lg H(t))^{\cdot\cdot} \geq 8$$

and

$$H(t) \leq H(0)^{1-t} H(1)^t$$

- Therefore u has a gaussian decay for $0 < t < 1$!!!

Remark.– If u_0 has gaussian decay then $e^{it\Delta}u_0$ does not necessarily have it for $t > 0$ $\left(u_0 = (\operatorname{sig} x) e^{-|x|^2}\right)$.

MUCHAS GRACIAS POR SU ATENCIÓN

Theorem 5.–

$$H(t) = \int_{\mathbb{R}^n} |u(x,t)|^2 e^{a(t)|x|^2}\,dx < C\big(H(0) + H(1)\big).$$

with

$$a(t) = \frac{R}{4\big(1 + R^2(t-1/2)^2\big)} \quad \text{for some}\, R > 0,$$

and

$$a(0) = \frac{R}{(4+R^2)} = \mu \leq \frac{1}{4}!!.$$

Above R is the smallest R such that $\frac{R}{(4+R^2)} = \mu$.

Misleading algebraic manipulations

Define

$$H(t) = \left\langle e^{a(t)\frac{|x|^2}{2}} u, u \right\rangle \qquad t \in [-1, 1]$$

Then H is (formally) $1/a$–log convex if a solves

$$(*) \qquad \ddot{a} - \frac{3}{2}\frac{\dot{a}^2}{a} + 32a^3 = 0.$$

If $a(x)$ solves $(*)$ then $a_R(x) = Ra(Rx)$ is also a solution. This easily leads to a contradiction !!!

Further results

- Morgan's theorem (Bonami, Demange, Jaming).
- Blow up profiles for non–linear dispersive equations (Meshkov counterexample).
- Sharp version of Hardy's theorem (joint work with M. Cowling).
- Paley–Wiener theorem.

Non-Linear Schrödinger Equation

Theorem 4.– $u_1, u_2 \in \mathcal{C}\left([0,T] \,:\, H^k(\mathbb{R}^n)\right)$ strong solutions of

$$\partial_t u = i\left(\Delta u + F(u,\overline{u})\right) \qquad x \in \mathbb{R}^n \quad t \in [0,T].$$

$$k > \frac{n}{2} \quad ; \quad F \text{ regular} \quad ; \quad F(0) = \partial_u F(0) = \partial_{\overline{u}} F(0) = 0.$$

Assume

$$e^{\frac{|x|^2}{\beta^2}}\left(u_1(0) - u_2(0)\right) \in L^2,$$

$$e^{\frac{|x|^2}{\alpha^2}}\left(u_1(T) - u_2(T)\right) \in L^2,$$

$$\alpha\beta < 4T.$$

Then $u_1 \equiv u_2$.