An inverse scattering problem in random media

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Outline

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The goal of inverse scattering

- The aim is to determine an electric potential q in $\mathbb{R}^n$ with $n \geq 2$ from scattering measurements.
- Scattering measurements encode information about how the potential scatterers certain incoming waves.
- To pose the inverse problem, one assumes some *a priori knowledge* in the **micro-structure** of the potential: q belongs to L^∞ or $L^{n/2}$, has compact support or certain decay at infinity.
- Sometimes the scatterer is so rough and complicated—lack of pattern in the micro-structure—that we need to assume the potential to be a **physical random process**.
- In these situations, the goal is not to recover the potential but to determine some functions describing properties of the micro-structure. For example: **the local strength** of a potential q.

Mathematical formulation

Consider the scattering problem

$$\begin{cases} (\Delta - q + k^2)u = 0 & \text{in } \mathbb{R}^n \\ u(x) = e^{ik\theta\cdot x} + u_{sc}(x) & \\ u_{sc} \text{ satisfies the Sommerfeld radiation condition,} & \end{cases}$$

- incident wave is $e^{ik\theta\cdot x}$ and, u_{sc} and u are the scattered and total waves.
- The scattered wave satisfies

$$u_{sc}(x) = c_n k^{\frac{n-1}{2}} |x|^{-\frac{n-1}{2}} e^{ik|x|} u^{\infty}\left(k, \theta, \frac{x}{|x|}\right) + o\left(|x|^{-\frac{n-1}{2}}\right),$$

  where u^{∞} is far-field pattern of u_{sc}.
- Inverse backscattering problem:

  (Q) *Given $u^{\infty}(k, \theta, -\theta)$ for $k > 0$ and $\theta \in \mathbb{S}^{n-1}$, can we recover q?*

Some references on deterministic random scattering

The literature on the deterministic inverse backscattering problem is considerably wide:

- Some authors with relatives works: Eskin–Ralston, Lagergren, Melrose–Uhlmann and Stefanov.
- Recovering singularities: Greenleaf–Uhlmann, Ola–Päivärinta–Serov, Reyes(–Ruiz) and Ruiz–Vargas.
- Uniqueness for angularly controlled potentials: Rakesh–Uhlmann.

The uniqueness for the inverse backscattering problem remains open.

Comments on random scattering

- We assume the potential $\omega \in \Omega \mapsto q(\omega)$ to be a generalized random function in a probability space $(\Omega, \mathcal{H}, \mathbb{P})$.
- This makes the **far-field pattern** be **random**—it changes with each realization $q(\omega)$.
- The problem consists of assuming $u^\infty(k, \theta, -\theta)$ with $k > 0$ and $\theta \in \mathbb{S}^{n-1}$ to be generated by a **single realization** $q(\omega_0)$, and then to determine the parameters characterizing the probability law of q.
- We reconstruct **the local strength** of q which is one of the parameters describing the probability law of q.

Remark

In the applied literature, the measure data is often assumed to be the averaged data, which corresponds to assume that the data has been generated by many independent samples of the scatterers. This is not justified if the scatterer does not change during the measurements.

Our data consists of $\{M(\tau_j, \theta_j) : j \in \mathbb{N}\}$ with

$$M(\tau, \theta) = \lim_{K \to \infty} \frac{1}{K} \int_K^{2K} k^m u^\infty(k, \theta, -\theta) \overline{u^\infty(k + \tau, \theta, -\theta)} dk.$$

Microlocally isotropic random potentials

A microlocally isotropic random potentials satisfies:

- The potential q is a **generalized Gaussian field** and is supported almost surely in bounded domain $D \subset \mathbb{R}^n$.
- Assume $\mathbb{E}q$ to be smooth and its covariance operator C_q, defined by

$$\langle C_q \phi, \psi \rangle = \mathbb{E}(\langle q - \mathbb{E}q, \phi \rangle \langle q - \mathbb{E}q, \psi \rangle),$$

  to be a **classical pseudodifferential operator** of order $-m$ with $n - 1 < m \leq n + 1$.
- C_q has a principal symbol of the form

$$\mu(x)|\xi|^{-m}$$

  with $\mu \in C_0^\infty(\mathbb{R}^n; \mathbb{R})$, $\operatorname{supp} \mu \subset D$ and $\mu(x) \geq 0$ for all $x \in \mathbb{R}^n$.
- The non-negative smooth function μ is called the local strength of q.

Example of microlocally isotropic random potential

Let X_H be a fractional Brownian motion in $\mathbb{R}^2$ with Hurst index $H \in (0,1)$, which means that X_H is a centered Gaussian field satisfying

$$\mathbb{E}|X_H(x_1) - X_H(x_2)|^2 = |x_1 - x_2|^{2H} \quad \forall x_1, x_2 \in \mathbb{R}^2,$$
$$X_H(x_0) = 0,$$
the paths $x \mapsto X_H(x)$ are a.s. continuous.

Example (Lassas, Päivärinta and Saksman)

Our example is given for $n = 2$ and $m = 2 + 2H$ by

$$q(x, \omega) = a(x) X_H(x, \omega)$$

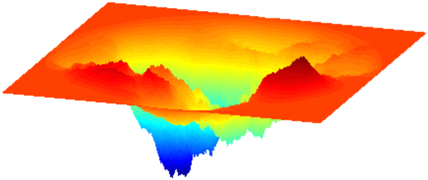
for a real-valued function $a \in C_0^\infty(\mathbb{R}^2)$ and $H \in (0, 1/2]$.

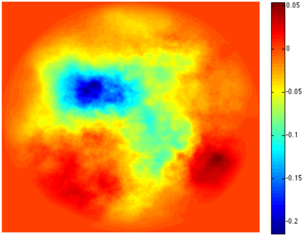
- The covariance operator C_q has

$$c_H a(x)^2 |\xi|^{-2-2H}$$

as principal symbol. Note how a **modulates the size** of q.

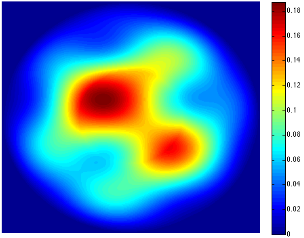
Comparing an arbitrary realization with its local strength





Top: Fractional Brownian motion with Hurst index $H = 0.8$

Right: The local strength $a(x)^2$



Properties of microllocaly isotropic random potential

- Since C_q is a pseudodifferential operator, its Schwartz kernel $K_q(x, y)$ is smooth out of the diagonal:

$$K_q(x,y) = \mathbb{E}\big((q(x) - \mathbb{E}q(x))(q(y) - \mathbb{E}q(y))\big).$$

- This means that the long distance interactions depends smoothly on their locations.
- The assumptions on C_q impose properties for q:

$$(\mathrm{I} - \Delta)^{s/2} q \in L^p_{\mathrm{loc}}(\mathbb{R}^n) \quad \text{a.s.}$$

  for $1 \leq p < \infty$ and $s < (m-n)/2$ with $n - 1 < m \leq n + 1$.
- This means that the **roughness** (or smoothness) of q remains unchanged for every sub-domain of D.

Interpretation of the local strength of q

- For every compact $K \subset \mathbb{R}^n$, there exists a $C > 0$ such that

$$\mathbb{E}\|(\mathrm{I} - \Delta)^{s/2}(q - \mathbb{E}q)\|_{L^p(K)} \leq C \sup_{x \in K} \mu(x).$$

- The **size of this average roughness changes** in different sub-domains of D. The local strength of the potential controls these different sizes.
- μ **yields valuable control on the oscillations of** q: where μ is small, the rough oscillations of q are most likely small as well.

Outline

Main theorem

Consider $\tau \in \mathbb{R}_+$ and $\theta \in \mathbb{S}^2$, then

$$M(\tau, \theta) = \lim_{K \to \infty} \frac{1}{K} \int_K^{2K} k^m u^\infty(k, \theta, -\theta) \overline{u^\infty(k + \tau, \theta, -\theta)} dk.$$

Theorem (C, Helin and Lassas)

Let q be a Gaussian microlocally isotropic random field of order -3 in $D \subset \mathbb{R}^3$. Then, the measurement data $\{M(\tau_j, \theta_j) : j \in \mathbb{N}\}$, with $\{(\tau_j, \theta_j) : j \in \mathbb{N}\}$ any dense subset of $\mathbb{R}_+ \times \mathbb{S}^2$, determines the local strength μ almost surely.

Remark

Our work also has consequence in general dimensions, but for simplicity we only state the case $n = m = 3$.

Previous results

Our result is inspired by an outstanding work of Lassas, Päivärinta and Saksman.

- They consider a similar problem in $\mathbb{R}^2$ for a backscattering problem with point sources in an open and bounded set.
- They assume knowledge of the full scattered wave.
- We study scattering of plane waves and only assume knowledge of the far-field patter of the backscattered wave.

Other results:

- Another previous result in this line is due to Helin, Lassas and Päivärinta: backscattering from random Robin boundary condition in a half-space of $\mathbb{R}^3$.
- The stochastic inverse problem has been considered by Bal and Jing in homogenization framework.

Outline

The Born series

We explore

$$M(\tau,\theta) = \lim_{K\to\infty} \frac{1}{K} \int_K^{2K} k^m u^\infty(k,\theta,-\theta)\overline{u^\infty(k+\tau,\theta,-\theta)}dk$$

by separating the different orders of scattering in the **Born series**

$$u^\infty(k,\theta,-\theta) = \sum_{j=1}^{\infty} u_j^\infty(k,\theta,-\theta).$$

It can be shown that the **simple backscattering**

$$\lim_{K\to\infty} \frac{1}{K} \int_K^{2K} k^m u_1^\infty(k,\theta,-\theta)\overline{u_1^\infty(k+\tau,\theta,-\theta)}dk = c\,\widehat{\mu}(2\tau\theta) \quad \text{a.s.}$$

for a known constant. Moreover, the contribution from **double** and **multiple backscattering** becomes negligible almost surely

$$\lim_{K\to\infty} \frac{1}{K} \int_K^{2K} k^m u_j^\infty(k,\theta,-\theta)\overline{u_l^\infty(k+\tau,\theta,-\theta)}dk = 0 \quad \text{a.s. for } j+l \geq 3.$$

Some comments on the different orders

- In order to prove that the **single scattering** provides information about the Fourier transform of μ, we need to use some properties of Gaussian fields and an ergodicity theorem.
- Checking that the **double scattering** is negligible is the hardest part and a number of tools have to be used: restriction of the Fourier transform to the sphere, Isserlis' theorem and a detailed description of K_q.
- We see that the **multiple scattering** is negligible using estimates for the forward scattering. However, these estimates have to be proved because the realizations of q only belongs to $L^p_{-s}(\mathbb{R}^n)$ for all $1 < p < \infty$ and $s > 0$.
- We solved the **forward scattering** for compactly supported realizations of $q \in L^p_{-s}(\mathbb{R}^n)$ with $0 \leq s \leq 1/2$ and $p \geq n/s$—inspired by previous works of Agmon–Hörmander and Kenig–Ponce–Vega.

Single backscattering in the case $\mathbb{E}q = 0$

Recall that the interactions of the single backscattering were encoded in

$$\lim_{K\to\infty} \frac{1}{K} \int_K^{2K} k^m u_1^\infty(k,\theta,-\theta)\overline{u_1^\infty(k+\tau,\theta,-\theta)}dk$$

and we wanted to show that this limit converges to

$$c\,\widehat{\mu}(2\tau\theta) \quad \text{almost surely.}$$

- **We actually prove this for** $m > n-1$ **with** $n \geq 2$ using the following ergodicity property: if $\mathbb{E}X_t = 0$ and

  $$|\mathbb{E}(X_t X_{t+r})| \leq c(1+r)^{-\epsilon} \qquad r,t \geq 0,$$

  then

  $$\lim_{T\to\infty} \frac{1}{T} \int_T^{2T} X_t\, dt = 0 \quad \text{almost surely.}$$

Almost uncorrelated measurements

On the other hand, **using some properties of Gaussian random fields** we see that

$$|\mathbb{E}(u_1^\infty(k_1,\theta,-\theta)\overline{u_1^\infty(k_2,\theta,-\theta)})| \lesssim k_1^{-m}(1+|k_1-k_2|)^{-N}.$$

Therefore,

$$\begin{aligned}&\lim_{K\to\infty}\frac{1}{K}\int_K^{2K} k^m u_1^\infty(k,\theta,-\theta)\overline{u_1^\infty(k+\tau,\theta,-\theta)}\,dk\\ &\quad=\lim_{K\to\infty}\frac{1}{K}\int_K^{2K} k^m\mathbb{E}(u_1^\infty(k,\theta,-\theta)\overline{u_1^\infty(k+\tau,\theta,-\theta)})\,dk\end{aligned}$$

almost surely.

The limit

Note that

$$\begin{aligned}\mathbb{E}(u_1^\infty(k,\theta,-\theta)\overline{u_1^\infty(k+\tau,\theta,-\theta)}) &= \mathbb{E}(\langle q, e^{i2k\theta\cdot y}\rangle\overline{\langle q, e^{i2(k+\tau)\theta\cdot x}\rangle}) \\ &= \int_{\mathbb{R}^n}\left(\int_{\mathbb{R}^n} K_q(x,y)e^{-i2k\theta\cdot(x-y)}\,dy\right)e^{-i2\tau\theta\cdot x}\,dx \\ &= c_n\int_{\mathbb{R}^n}\mu(x)|2k|^{-m}e^{-i2\tau\theta\cdot x}\,dx + c_n\int_D a(x,2k\theta)e^{-i2\tau\theta\cdot x}\,dx.\end{aligned}$$

Here we have used the well known relation between the symbol of C_q, $c_q(x,\xi)$ and its corresponding Schwartz kernel K_q

$$K_q(x,y) = \frac{1}{(2\pi)^{n/2}}\mathcal{F}^{-1}(c_q(x,\cdot))(x-y);$$

and the fact that C_q has $\mu(x)|\xi|^{-m}$ as a **principal symbol**.
Eventually,

$$\lim_{K\to\infty}\frac{1}{K}\int_K^{2K}k^m u_1^\infty(k,\theta,-\theta)\overline{u_1^\infty(k+\tau,\theta,-\theta)}dk = c_{n,m}\,\widehat{\mu}(2\tau\theta) \quad \text{a.s.}$$

Outline

To sum up

- In **random scattering** the goal is not to recover the potential but to determine some functions describing properties of the micro-structure.
- Here we have reconstructed the local strength of a potential q, which is **the principal symbol of its covariance operator** and controls locally the oscillations of q.
- Our measurements have been generated by a **single realization** of the potential.