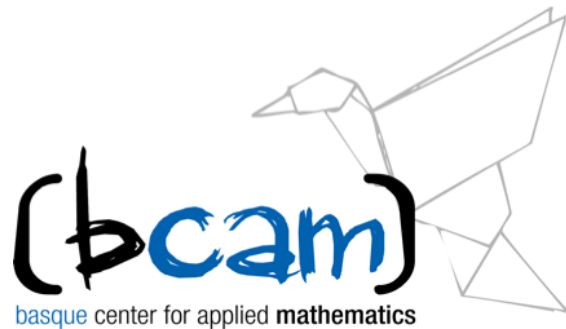


LOWER BOUNDS FOR OSCILLATORY INTEGRALS

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- (1) Uncertainty principles (with **L. Escauriaza**, **C. Kenig** and **G. Ponce**)
- (2) A space time lower bound (**EKPV**)
- (3) New results (with **M. Agirre**)
 - A lower bound for a fixed time
 - Uniqueness
- (4) About the proof.

Hardy's Uncertainty Principle (Cowling, Price)

Define for $\xi \in \mathbb{R}^n$

$$\widehat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} f(x) dx$$

Assume that

$$f(x) e^{\frac{|x|^2}{\alpha^2}} \in L^2$$

$$\widehat{f}(\xi) e^{\frac{|\xi|^2}{\beta^2}} \in L^2$$

$$\alpha^2 \beta^2 \leq 4.$$

Then

$$f \equiv 0$$

Morgan and Paley-Wiener

Morgan

If

$$\int |f|^2 e^{-|x|^p} dx < +\infty$$
$$\int |\widehat{f}|^2 e^{-|\xi|^{p'}} d\xi < +\infty$$
$$1 < p < \infty \quad \frac{1}{p} + \frac{1}{p'} = 1$$

then $u \equiv 0$.

Paley-Wiener

If

$$\int |f(x)| e^{-\epsilon|x|} dx < +\infty, \quad \epsilon > 0,$$

then \widehat{f} can not have compact support.

Free Schrödinger Equation

$$\text{S.E.} \quad \begin{cases} \partial_t u &= i\Delta u & x \in \mathbb{R}^n & t \in \mathbb{R} \\ u(0) &= u_0. \end{cases}$$

$$\begin{aligned} u(x, t) &= \frac{1}{(it)^{n/2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y) dy \\ &= \frac{1}{(it)^{n/2}} e^{i\frac{|x|^2}{4t}} \int_{\mathbb{R}^n} e^{-i\frac{x}{2t} \cdot y} e^{i\frac{|y|^2}{4t}} u_0(y) dy \end{aligned}$$

$$f(y) = e^{i\frac{|y|^2}{4t}} u_0(y),$$

$$\hat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} f(y) dy.$$

Hence

$$\begin{aligned} u(0)e^{\frac{|y|^2}{\alpha^2}} \in L^2 &\iff f e^{\frac{|y|^2}{\alpha^2}} \in L^2 \\ u(T)e^{\frac{|x|^2}{\beta^2}} \in L^2 &\iff \hat{f}\left(\frac{x}{2T}\right) e^{\frac{|x|^2}{\beta^2}} \in L^2. \end{aligned}$$

Hardy's uncertainty principle:

$$\alpha\beta \leq 4T \quad \text{then } u \equiv 0$$

Time/Energy uncertainty principle (Wave Packets)

$$u(x, t) = \frac{1}{(t+i)^{n/2}} e^{-\frac{|x|^2}{4(t+i)}}$$

$$|u|^2 = \frac{1}{(t^2+1)^{n/2}} e^{-\frac{|x|^2}{2(t^2+1)}}$$

- $u_R(x, t) = u(Rx, R^2t)$ is also a solution.
- $u^{t_0}(x, t) = u(x, t - t_0)$ is also a solution.

Schrödinger Equation with a Potential

$$u = u(x, t)$$

$$\partial_t u = i(\Delta + V)u \quad x \in \mathbb{R}^n$$

$$\left. \begin{array}{l} \bullet e^{\frac{|x|^2}{\alpha^2}} u(0) \in L^2 \\ \bullet e^{\frac{|x|^2}{\beta^2}} u(T) \in L^2 \\ \bullet \alpha\beta \text{ small enough} \end{array} \right\} \implies u \equiv 0$$

With C.E. Kenig, L. Escauriaza and G. Ponce

Theorem.— $u \in \mathcal{C}([0, T] : L^2(\mathbb{R}^n))$ solution of

$$\partial_t u = i(\Delta + V)u \quad x \in \mathbb{R}^n \quad t \in [0, T].$$

$$u(0)e^{\frac{|x|^2}{\alpha^2}} \in L^2 \quad ; \quad u(T)e^{\frac{|x|^2}{\beta^2}} \in L^2,$$

and $\alpha\beta < 4T$, then $u \equiv 0$.

Hypothesis of the potential:

H1 $V = V_0(x) +$ perturbation with a gaussian decay
 V_0 real and bounded

H1* $V = V(x, t)$
$$\lim_{R \rightarrow \infty} \int_0^T \sup_{|x| \geq R} |V| dt = 0$$

Nazarov's Uncertainty Principle

If E and F are subsets of \mathbb{R} with finite Lebesgue measure $|E|$, $|F|$ and $\text{supp} f \subset E$ then there exists a constant C such that

$$\int_F |\hat{f}|^2 dx < C e^{C|E||F|} \int_{\mathbb{R} \setminus F} |\hat{f}|^2 dx.$$

This statement is equivalent to the following one. Assume E and F as before and that u is a solution of the 1d Schrödinger equation, with support on $t = 0$ contained in the set E . Then

$$\int_F |u(x, 1)|^2 dx < C e^{C|E||F|} \int_{\mathbb{R} \setminus F} |u(x, 1)|^2 dx.$$

Q.— Can this result be obtained using Carleman estimates?

About the proofs in the constant coefficient case (key words):

▶ Uncertainty principle:

- Positive Commutators.

▶ Hardy's theorem:

- Analyticity, Cauchy–Riemann equations,
Log convexity.
- Liouville's theorem.

Log-Convexity (a dynamic uncertainty principle)

S symmetric; \mathcal{A} antisymmetric:

$$\partial_t v = (S + \mathcal{A})v$$

$$H(t) = \langle v, v \rangle$$

$$\begin{aligned}\dot{H}(t) &= \langle v_t, v \rangle + \langle v, v_t \rangle \\ &= \langle (S + \mathcal{A})v, v \rangle + \langle v, (S + \mathcal{A})v \rangle \\ &= 2\langle Sv, v \rangle.\end{aligned}$$

$$\begin{aligned}\ddot{H}(t) &= 2\langle Sv_t, v \rangle + 2\langle Sv, v_t \rangle \\ &= 2\langle (S + \mathcal{A})v, Sv \rangle + 2\langle Sv, (\mathcal{A} + S)v \rangle \\ &= 4\langle Sv, Sv \rangle + 2\langle (S\mathcal{A} - \mathcal{A}S)v, v \rangle.\end{aligned}$$

$$\begin{aligned}
(\lg H(t))'' &= \left(\frac{\dot{H}}{H} \right)' = \frac{\ddot{H}H - \dot{H}^2}{H^2} \\
&= \frac{1}{\langle v, v \rangle} \{ 4\langle Sv, Sv \rangle \langle v, v \rangle - 4\langle Sv, v \rangle^2 + 2 \langle (S\mathcal{A} - \mathcal{A}S)v, v \rangle \} \\
&\geq 2 \langle (S\mathcal{A} - \mathcal{A}S)v, v \rangle
\end{aligned}$$

Hence if $2 \langle (S\mathcal{A} - \mathcal{A}S)v, v \rangle \geq 0$ then

$$H(t) \leq H(0)^{1-t} H(1)^t$$

More generally, if

$$2 \langle (S\mathcal{A} - \mathcal{A}S)v, v \rangle \geq \psi(t) \langle v, v \rangle,$$

then

$$H(t)e^{-B(t)} \leq H(0)^{1-t} H(1)^t,$$

with

$$\ddot{B} = \Psi(t), \quad B(0) = B(1) = 0.$$

A Particular Example

- $\partial_t u = i\Delta u$

- $e^{\frac{|x|^2}{2}} u = v$

- $H(t) = \|v(t)\|_{L^2}^2 = \langle v(t), v(t) \rangle$

- $$\begin{aligned}\partial_t v &= \left(e^{\frac{|x|^2}{2}} i\Delta e^{-\frac{|x|^2}{2}} \right) v \\ &= i \sum_j e^{\frac{|x|^2}{2}} \partial_j^2 e^{-\frac{|x|^2}{2}} v \\ &= i \sum_j e^{\frac{|x|^2}{2}} \partial_j e^{-\frac{|x|^2}{2}} e^{\frac{|x|^2}{2}} \partial_j e^{-\frac{|x|^2}{2}} v \\ &= i \sum_j (x_j - \partial_j)(x_j - \partial_j) v \\ &= i(|x|^2 - 2x \cdot \nabla - 1 + \Delta) v\end{aligned}$$

(C-R)

$$\partial_t v = (S + \mathcal{A})v \quad ; \quad Sv = -i(2x \cdot \nabla + 1)v \quad ; \quad \mathcal{A}v = i(\Delta + |x|^2)v$$

$$S\mathcal{A} - \mathcal{A}S = -4\Delta + 4|x|^2$$

$$\langle S\mathcal{A} - \mathcal{A}Sv, v \rangle = 4(\|\nabla v\|^2 + \|xv\|^2) \geq 4\langle v, v \rangle.$$

Hence

$$(\lg H(t))'' \geq 8$$

and

$$H(t) \leq H(0)^{1-t} H(1)^t$$

- Therefore u has a gaussian decay for $0 < t < 1$!!

Remark.— If u_0 has gaussian decay, then $e^{it\Delta}u_0$ does not necessarily have it for $t > 0$ ($u_0 = ((\text{sig } x)e^{-|x|^2})$).

A space time lower bound

Theorem (with L. Escauriaza, C. Kenig and G. Ponce).–

Let $u \in \mathcal{C}([0, 1] : H^1(\mathbb{R}^n))$ be a strong solution of

$$i\partial_t u + \Delta u + Vu = 0, \quad t \in [0, 1], \quad x \in \mathbb{R}^n$$

If

$$\int_0^1 \int_{\mathbb{R}^n} (|u|^2 + |\nabla_x u|^2)(x, t) dx dt \leq A^2,$$

$$\int_{1/2-1/8}^{1/2+1/8} \int_{|x|<1} |u|^2(x, t) dx dt \geq 1,$$

and

$$\|V\|_{L^\infty(\mathbb{R}^n \times [0, 1])} \leq L,$$

then there exists $R_0 = R_0(n, A, L) > 0$ and a constant $c = c(n)$ such that for $R \geq R_0$ it follows that

$$\delta(R) \equiv \left(\int_0^1 \int_{R-1 < |x| < R} (|u|^2 + |\nabla_x u|^2)(x, t) dx dt \right)^{1/2} \geq ce^{-cR^2}.$$

δ -wave packets

$$\begin{cases} \partial_t u = \frac{i}{2} \Delta u & x \in \mathbb{R}^n \\ u(x, 0) = f(x), \end{cases}$$

We measure regularity using the space

$$\Sigma_\delta(\mathbb{R}^n) := \left\{ f \in L^2(\mathbb{R}^n) \mid \|f\|_{\Sigma_\delta}^2 := \||x|^\delta f\|_2^2 + \|D^\delta f\|_2^2 < \infty \right\},$$

where $D^\delta f := |\xi|^\delta \hat{f}(\xi)$

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-2\pi i x \xi} f(x) dx.$$

$$h_\delta(t) = \int |x|^{2\delta} |u(x, t)|^2 dx \quad 0 < \delta \leq 1$$

- Upper bound: persistence

$$\|e^{i|x|^2} f\|_{\Sigma_\delta} \leq c_+ \|f\|_{\Sigma_\delta} \quad (\text{Nahas-Ponce 2009})$$

Scaling gives

$$h_\delta(t) \leq c_+ (1 + t^2)^\delta$$

Theorem (with M. Aguirre).— Let $u \in \mathcal{C}([0, 1] : H_{\text{loc}}^1(\mathbb{R}^n))$ be a solution of

$$\begin{cases} \partial_t u = i(\Delta u + V(x, t)u) \\ u(x, 0) = u_0(x) \end{cases}$$

where $V \in L^\infty(\mathbb{R}^n \times [0, 1])$ is a complex potential and

$$\|V\|_{L^\infty(\mathbb{R}^n \times [0, 1])} \leq L$$

Let $R_1 \geq R_0 > 0$ be such that for some $c_0 > 0$,

$$\int_{B_{R_0}} |u_0|^2 dx = c_0^2,$$

and

$$\sup_{0 \leq t \leq 1} \int_{B_{R_1}} (|u(x, t)|^2 + |\nabla u(x, t)|^2) dx = A^2 < +\infty.$$

Then there exists $t^* = \min\left(R_0^2, L^{-2}, \frac{A}{8c_0L}, \left(\frac{2c_0^2}{16A^2}\right)^2\right)$ such that if $0 < t < t^*$ and $\rho \geq R_0$

$$\frac{e^{c(n)\frac{\rho^2}{t}}}{t} \int_{t/2}^t \int_{\rho(1+s/t) \leq |y| \leq \rho(1+s/t) + \sqrt{t}} (|u(y, s)|^2 + s|\nabla_y u(y, s)|^2) dy ds \geq \frac{c_0^2}{4}$$

Theorem.— Assume that for any $u_0 \in H^1(\mathbb{R}^n)$ there exists a unique solution $u \in \mathcal{C}([0, 1] : H^1(\mathbb{R}^n))$ of

$$\begin{cases} \partial_t u = i(\Delta + V(x, t))u & x \in \mathbb{R}^n, \quad t \in (0, 1) \\ u(x, 0) = u_0, \end{cases}$$

with $V \in L^\infty(\mathbb{R}^n \times [0, 1])$.

If there exists $R_j \rightarrow \infty$, $j \in \mathbb{N}$ such that for some u

$$\lim_{t \downarrow 0} \frac{1}{t} e^{c(n) \frac{R_j^2}{t}} \int_{t/2}^t \int_{R_j(1+s/t) \leq |y| \leq R_j(1+s/t) + \sqrt{t}} |u(x, s)|^2 + s |\nabla u(x, s)|^2 dx ds = 0,$$

then $u \equiv 0$.

About the proof

Lemma.– If $u(y, s)$ verifies

$$\partial_s u = i(\Delta u + V(y, s)u + F(y, s)), \quad (y, s) \in \mathbb{R}^n \times [0, 1]$$

and α and β are positive, then

$$\tilde{u}(x, t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t) + \beta t} \right)^{n/2} u \left(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t) + \beta t}, \frac{\beta t}{\alpha(1-t) + \beta t} \right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t) + \beta t)}}$$

verifies

$$\partial_t \tilde{u} = i(\Delta \tilde{u} + \tilde{V}(x, t)\tilde{u} + \tilde{F}(x, t)), \quad (x, t) \in \mathbb{R}^n \times [0, 1]$$

with

$$\tilde{V}(x, t) = \frac{\alpha\beta}{(\alpha(1-t) + \beta t)^2} V \left(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t) + \beta t}, \frac{\beta t}{\alpha(1-t) + \beta t} \right)$$

and

$$\tilde{F}(x, t) = \left(\frac{\sqrt{\alpha\beta}}{\alpha(1-t) + \beta t} \right)^{n/2+2} F \left(\frac{\sqrt{\alpha\beta}x}{\alpha(1-t) + \beta t}, \frac{\beta t}{\alpha(1-t) + \beta t} \right) e^{\frac{(\alpha-\beta)|x|^2}{4i(\alpha(1-t) + \beta t)}}$$

Lemma (Carleman estimate).— Assume that $R > 0$ and $\varphi : [0, 1] \rightarrow \mathbb{R}$ is a smooth function. Then, there exists $c(n) = c(n, \|\varphi'\|_\infty + \|\varphi''\|_\infty) > 0$ such that the inequality

$$\frac{\sigma^{3/2}}{c(n)R^2} \|e^{\sigma|\frac{x}{R} + \varphi(t)e_1}|^2 g\|_2 \leq \|e^{\sigma|\frac{x}{R} + \varphi(t)e_1}|^2 (i\partial_t + \Delta)g\|_2$$

holds when $\sigma \geq c(n)R^2$ and $g \in \mathcal{C}_0^\infty(\mathbb{R}^{n+1})$ has its support contained in the set

$$\left\{ (x, t) : \left| \frac{x}{R} + \varphi(t)e_1 \right| \geq 1 \right\}$$

The goal is to use Carleman's estimate (referencia) in a suitable way so that we can control both u and ∇u by the initial data. For this purpose we want to build a function g with specific functions. First, let $\gamma > 1$ and define $R = R_0\sqrt{\gamma}$. Define also the following cut-off functions, $\theta_R, \eta \in \mathcal{C}_0^\infty(\mathbb{R}^n)$, $\varphi \in \mathcal{C}^\infty([0, 1])$

$$\theta_R(x) = \begin{cases} 1, & |x| \leq R \\ 0, & |x| \geq R + 1 \end{cases} \quad \eta(x) = \begin{cases} 1, & |x| \geq 2 \\ 0, & |x| \leq 3/2 \end{cases}$$

$$\varphi(t) = \begin{cases} 4, & t \in [3/8, 5/8] \\ 0, & t \in [0, 1/4] \cup [3/4, 1] \end{cases}$$

We define the new unknown v ,

$$v(x, t) = \alpha(t)^{n/2} u(\alpha(t)x, s(t)) e^{-\frac{i}{2}\beta(t)|x|^2}, \quad (x, t) \in \mathbb{R}^n \times [0, 1]$$

Given γ , define α , β and s

$$\alpha(t) = \frac{1}{\gamma^{1/2}(1-t) + \gamma^{-1/2}t} \leq \frac{1}{\gamma^{1/2}(1-t)} \leq \frac{4}{\gamma^{1/2}}$$

$$\beta(t) = \frac{1}{1-t + \gamma^{-1}t} - \frac{1}{\gamma(1-t) + t} \leq \frac{1}{1-t + \gamma^{-1}t} \leq 4$$

$$\frac{s(t)}{t} = \frac{1}{\gamma(1-t) + t} \leq \frac{1}{\gamma(1-t)} \leq \frac{4}{\gamma}$$

We use all the information gathered above to define the function g as follows:

$$g(x, t) = \theta_R(x) \eta \left(\frac{x}{R} + \varphi(t) e_1 \right) v(x, t) \quad , \quad (x, t) \in \mathbb{R}^n \times [0, 1]$$

$$\mathbf{R} = \mathbf{R}_0 \gamma^{1/2}$$

Observe that due to the nature of the test functions, g is compactly supported and,

- $g = \theta_R v$ on $(x, t) \in \{|x| \leq R + 1\} \times [3, 8, 5/8]$
- $\nabla_x v(x, t) = \alpha(t)^{n/2} e^{-\frac{i}{2}\beta(t)|x|^2} (\alpha(t)\nabla u - i\beta(t)x \cdot u)$
- $\text{supp } g \subseteq \left\{ \left| \frac{x}{R} + \varphi e_1 \right| \geq 1 \right\}$.

**THANK YOU FOR YOUR
ATTENTION**