New Conservation Laws and Energy Cascade for 1d Cubic NLS

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Summary

$$u(x,t) = c_M \sum_k e^{itk^2+ikx}$$

- It is a "solution" of 1d–cubic NLS **(NLS)**
- It has a geometrical meaning:

  Binormal Flow **(BF)**

  Schrödinger map **(SM)**

- (NL) Talbot effect:
  - Intermittency
  - Multifractality

  } **Turbulence**

PDE' problem

- NLS
- NLS $\Longrightarrow$ SM: Linear system
- SM $\Longrightarrow$ BF: ODF. One trajectory:

  Riemann's non differentiable function

Example: $u = c_M \frac{1}{\sqrt{t}} e^{ix^2/4t}$

- Selfsimilar solution SM and BF

  Coherent Structure

We consider the IVP

$$\partial_t u = i\left(\partial_x^2 u \pm \left(|u|^2 - M(t)\right)u\right) \qquad M(t) \in \mathbb{R}$$
$$u(x,0) = u_0(x) \qquad x \in \mathbb{R}$$

- **Hasimoto transformation:**

  $+$: **Focusing** Vortex Filament Equation (VFE)

  Schrödinger Map (SM) onto $\mathbb{S}^2$

  $-$: **Defocusing** VFE (hyperbolic geometry)

  SM onto $\mathbb{H}^2$

- Data at the critical level of regularity:

(i) Scaling invariance:

$$\lambda > 0 \qquad u_\lambda(x,t) = \lambda u(\lambda x, \lambda^2 t)$$

(ii) Galilean invariance:

$$\nu \in \mathbb{R} \qquad u^\nu(x,t) = e^{-it\nu^2 + i\nu x} u(x - 2\nu t, t)$$

In the free case

$$e^{it\xi^2}\widehat{u}(\xi,t) = \widehat{u}_0(\xi)$$

What happens in the non–linear case?

- If $\omega(\xi,1) = e^{i\xi^2}\widehat{u}(\xi,1)$ is periodic then

  $\omega(\xi,t) = e^{it\xi^2}\widehat{u}(\xi,t)$ is also (formally) periodic

- Phase blow up **(BLTV, 2023)**:

  $\lim_{t\downarrow 0} |\widehat{\omega}(j,t)|^2 = |a_j|^2$ exists but

  $\lim_{t\downarrow 0} \widehat{\omega}(j,t)$ does not exist.

$$u(x,t) = \sum_j A_j(t) e^{it\partial_x^2} \delta(x-j)$$

$$\widehat{u}(\xi,t) = e^{-it\xi^2} \sum_j A_j(t) e^{ij\xi}$$

We define

$$V(\xi,t) = \sum_j B_j(t) e^{ij\xi}$$

Observe that

$$u(x,t) = \frac{1}{(it)^{1/2}} \sum_j A_j(t) e^{i\frac{(x-j)^2}{4t}}$$
$$= \frac{1}{(it)^{1/2}} e^{i\frac{|x|^2}{4t}} \sum_j A_j(t) e^{i\frac{j^2}{4t} - i\frac{x}{2t}j}$$
$$:= \frac{1}{(it)^{1/2}} e^{i\frac{|x|^2}{4t}} \overline{V}\left(\frac{x}{t}, \frac{1}{t}\right)$$

$$B_j(t) = A_j\left(\tfrac{1}{t}\right) e^{-i\frac{t}{4}j^2}$$

Hence $\xi = \dfrac{x}{t}$ **!!**

Moreover

$$\partial_t V = i\left(\partial_\xi^2 + \frac{1}{t}(|V|^2 - m^2)\right)V \qquad ; \qquad M(t) = \frac{m^2}{t}.$$

t becomes $1/t$

u_0 is the scattering data (i.e. $V \sim ? \quad t \to \infty$).

- **Remark**:

  - There is a singularity at $t = 0$. Hence a very natural question is if u can be continued for $t \leq 0$.

  - The phase loss problem: **Bourgain, Merle**.

Example 1

$V(1) = a$ Then the solution is

$$V(\xi, t) = a$$

- **Remark**: The corresponding solution is

$$u(x,t) = \frac{1}{\sqrt{t}} e^{i\frac{|x|^2}{4t}} e^{-ia^2 \lg t}$$

- **Remark**: IVP $u_0 = a\delta$ is ill–posed **(KPV 2001)**

Example 2 (Banica–V. 2008–2013)

$a \in \mathbb{R}$: $V = a + z$ with z small with respect to a.
Then, there exists a unique solution for $t \geq 1$ for $z(1)$ in a nice space (i.e. $\mathcal{C}_0^\infty(\mathbb{R})$).
Generically

$$\int_{-\infty}^{\infty} z(\xi, t) d\xi = c \lg t \qquad c \neq 0$$

Corollary $e^{it\xi^2}\widehat{u}(\xi, t) - a$ is not zero at infinity. There is a cascade to large frequencies for $t \to 0$.

The periodic case

We are given u_0 ; $\widehat{u}_0(\xi) = \sum_j a_j e^{ij\xi}$

We want to solve

$$\partial_t V = i\left(\partial_\xi^2 V + \frac{1}{t}(|V|^2 - m^2)V\right)$$

such that

$$V(\xi,t) = \sum_j B_j(t) e^{ij\xi}$$

$$A_j(t) = e^{it\frac{j^2}{4}} B_j(1/t)$$

$$A_j(0) = a_j \text{ ?}$$

The two conservation laws

(1) $\underline{m} = \int_0^{2\pi} |V(\xi,t)|^2 d\xi$ is constant for $t > 0$.

$\underline{m} = \sum_{j=1} |a_j|^2 \quad ; \quad l^2$–condition.

(2) If $a_{j+M} = a_j$ for some M, then

$\underline{m} = \sum_{j=1}^{M} |a_j|^2$ is constant for $t > 0 \quad ; \quad l^\infty$–condition.

For VFE at $t = 0$

(1) Open polygonal lines

(2) For properly choices of "a_j":

Closed polygons

- The curves are not necessarily planar because $a_j = \rho_j e^{i\tau_j}$

  ρ_j: the angle between two segments.

  τ_j: the torsion (angle between two consecutive planes.)

Theorems

(1) $a_j \in l^1$: No smallness

Global existence if $\sum_j j^2 |a_j|^2 < +\infty$ **(BV 2019)**

Local time of existence can be big.

Particular relevant example:

$a_j = 1 \quad \text{for} \quad |j| \leq N$ **(First iterate is "small")**

VFE: Riemann's non–differential function

Multifractal formalism Frisch–Parisi

(2) • $a_j \in l^2 \quad \|a_j\|_{l^\infty}$ small; global existence

• $\|a_j\|_{H^\epsilon} \quad \epsilon > 0$ Phase blow–up **(BLTV, 2023)**

(3) $a_j \in l^p \quad p < \infty$

• $\|a_j\|_{l^p}$ small

(Bravin–V. 2021)

• Time of existence small

(4) $a_j = a \quad \forall j$ explicit example **(Regular polygons)**

(Banica–Bravin–V. 2021)

• **Tsutsumi, Bourgain, Vargas–V, Christ–Colliander–Tao, Grunrock–Herr, Killip–Visan, Koch–Tataru, Kappeler**

Proofs

- Right choice of $M(t) = \dfrac{m^2}{t}$
- Ansatz

$$u(x,t) = \sum_j A_j(t) e^{i\phi_j(t)} e^{it\partial_x^2} \delta(x-j)$$

$$\phi_j(t) = e^{i\frac{|a_j|^2}{8\pi}} \lg t$$

- Infinite dynamical system for $\{A_j(t)\}_j$,

$$A_j = a_j + R_j(t)$$

Fixed point argument for $\{R_j(t)\}_j$

- **Corollary** The IVP for u at $t = 0$ is ill–posed.

Nevertheless there is a "unique" continuation for $t < 0$ if u is understood as the "filament function" associated to VFE and SM.

Resonances and cascade of energy for T

The equation for T (SM) is

$$T_t = T \wedge T_{xx}$$

$$\partial_t |T_x|^2 = \partial_x(\text{flux})$$

Hence $|T_x|^2 dx$ is a natural energy density.

Moreover

$$\int_0^{2\pi} |V(\xi,t)|^2 d\xi = \lim_{n\to\infty} \int_{2\pi n}^{2\pi(n+1)} \left|\widehat{T_x}(\xi,t)\right|^2 d\xi$$

Theorem **(BV 2021)**

Assume

$$\begin{cases} a_{-1} = a = a_{+1} & a \neq 0 \\ a_j = 0 & \text{otherwise} \end{cases}$$

Then there exists $c > 0$

$$\sup_{\xi} \left| \widehat{T_x}(\xi, t) \right|^2 \geq c |\lg t| \qquad t > 0.$$

- **Colliander, Keel, Staffilani, Takaoka, Tao 2010**
- **Hani, Pausader, Tzvetkov,Visciglia 2015**

- This cascade can be understood associated to a linear problem.

$$(T, e_1, e_2) \quad \text{orthonormal frame}$$

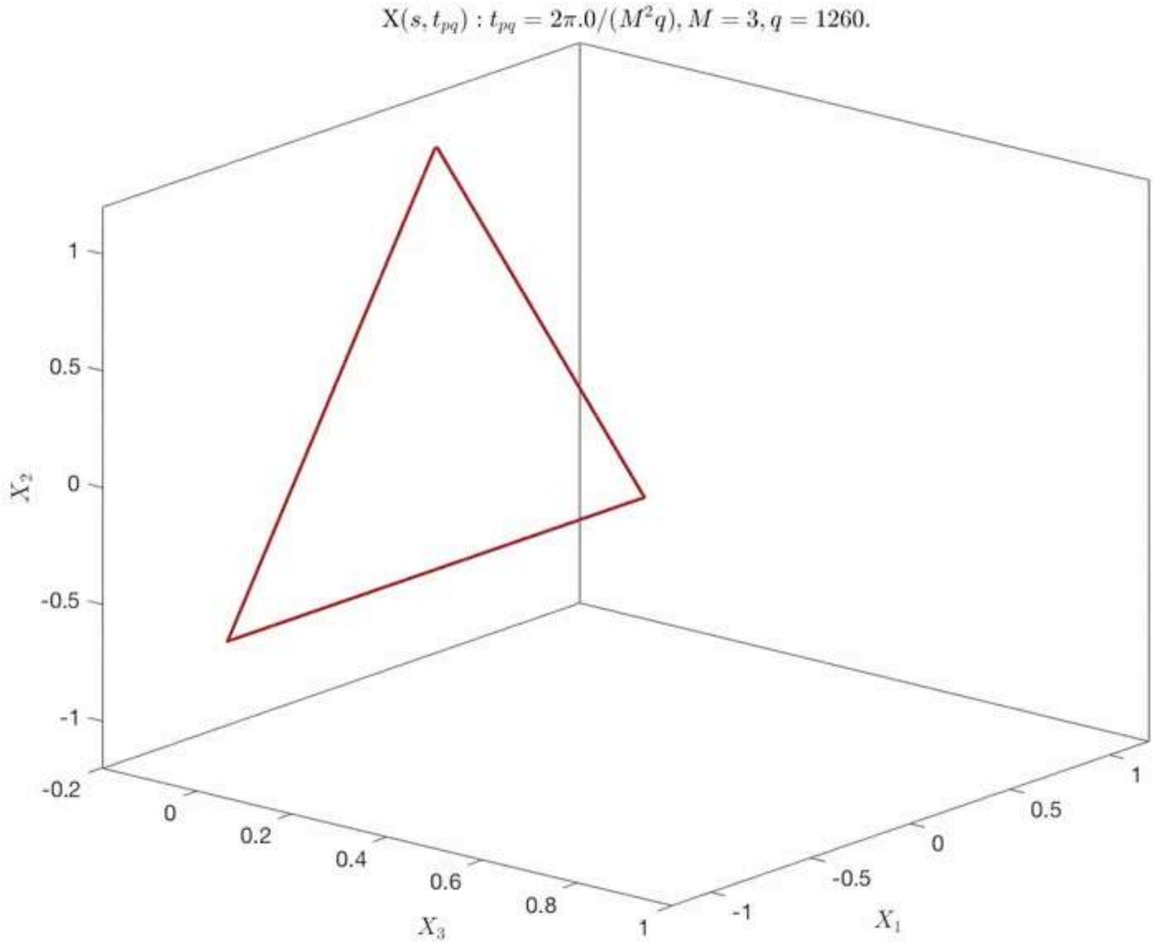
$$\begin{aligned} T_x &= \alpha e_1 + \beta e_2 \\ e_{1x} &= -\alpha T \\ e_{2x} &= -\beta T \end{aligned}$$

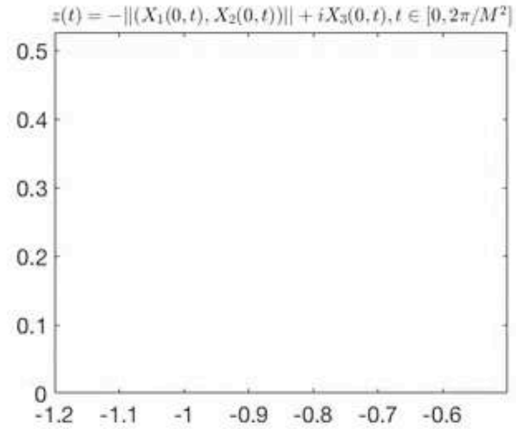
$$\begin{aligned} T_t &= -\beta_x e_1 + \alpha_x e_2 \\ e_{1t} &= -\alpha_x T + (|u|^2 - M(t)) e_2 \\ e_{2t} &= -\beta_x T - (|u|^2 - M(t)) e_1 \end{aligned}$$

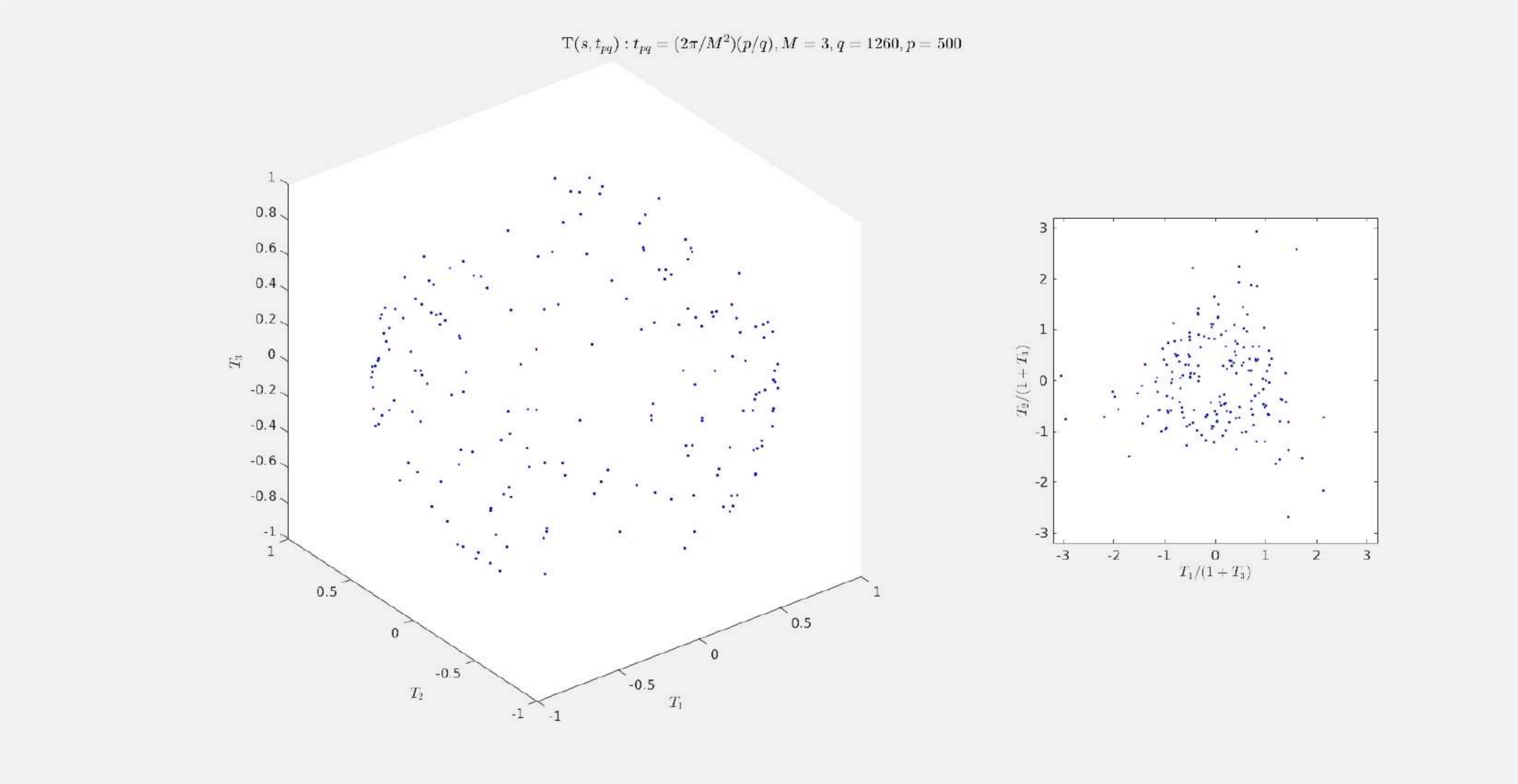
$$\alpha + i\beta = \frac{1}{\sqrt{t}} e^{i\frac{|x|^2}{4t}} \overline{V}\left(\frac{x}{2t}, \frac{1}{t}\right) = u(x,t)$$

$$\alpha_x + i\beta_x = u_x = i\frac{x}{2t}u + \text{“small”}$$

(Chevillard et al. 2021)


X$(s, t_{pq}) : t_{pq} = 2\pi.0/(M^2 q), M = 3, q = 1260.$
X_2
X_3
X_1
1
0.5
0
-0.5
-1
-0.2
0
0.2
0.4
0.6
0.8
1
-1
-0.5
0
0.5
1


$z(t) = -||(X_1(0,t), X_2(0,t))|| + iX_3(0,t), t \in [0, 2\pi/M^2]$
0.5
0.4
0.3
0.2
0.1
0
-1.2
-1.1
-1
-0.9
-0.8
-0.7
-0.6


$\mathrm{T}(s, t_{pq}) : t_{pq} = (2\pi/M^2)(p/q), M = 3, q = 1260, p = 500$
T_3
T_2
T_1
$T_2/(1+T_3)$
$T_1/(1+T_3)$

THANK YOU FOR YOUR ATTENTION