The binormal flow, the Talbot effect, and non-circular jets

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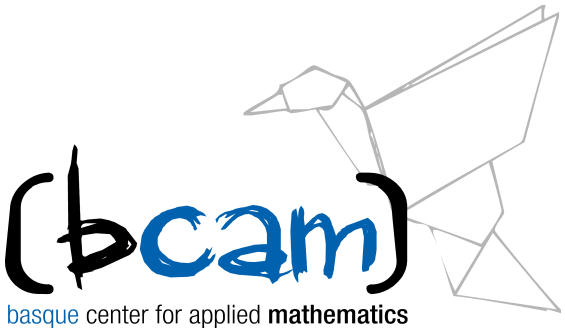
(with V. Banica and F. De la Hoz)



PDE zoom-seminar, Shanghai, Sept. 10, 2020

Summary

(A) Binormal Flow **(BF)** a dispersive (geometric) PDE

- Critical regularity: Polygonal lines
- Selfsimilar solutions have finite energy $\left(\widehat{L^2}_{\text{per}}\right)$
- Talbot effect
- Continuation beyond the singularity time

(B) **BF** as a toy model in Fluid Mechanics: Vortex filament equation **(VFE)**

- Coherent structures are the self similar solutions
- Interaction: A weakly non–linear Talbot effect **(NLTe)**

Conjecture: **NLTe** can explain the turbulent dynamics of non circular jets

Q: **BF** for regular Polygons and regular polygonal helices

(A) Binormal Flow

(BF) • $\chi_t = \chi_s \wedge \chi_{ss} = cb$ $\quad c$: curvature $\quad b$: binormal

(SM) • $\chi_s = T$ Schrödinger map $\qquad T_t = T \wedge T_{ss}$

(NLS) • ψ Hashimoto wave function 1d **NLS** (cubic focusing)

$\chi(0,s)$: skew polygonal line

$T(0,s)$: sequence of points T_j such that $\lim_{j\to\pm\infty} T_j = A^{\pm}$

$$\psi(0,s) = \sum a_j \delta(s-j) \qquad \sum_j |j|^{1^+} |a_j|^2 < +\infty$$

$$\begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}_s = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}$$

$$\psi = \alpha + i\beta$$

SCHRÖDINGER EQUATION

Hasimoto transformation:

$$\psi(s,t) = c(s,t)e^{i\int_0^s \tau(s',t)ds'}$$

$c = c(s,t)$ curvature

$\tau = \tau(s,t)$ torsion

$$\partial_t \psi(s,t) = i\left(\partial_s^2 \psi \pm \frac{1}{2}(|\psi|^2 + A(t))\psi\right)$$

$$\int_{-\infty}^{\infty} |\psi(s,t)|^2 ds = \int_{-\infty}^{\infty} |\psi(s,0)|^2 ds = \int_{-\infty}^{\infty} c^2(s,0) ds$$

In our case

$$\psi(s,t) = \frac{a}{\sqrt{t}} e^{i\frac{s^2}{4t}} \qquad , \qquad \int_{-\infty}^{\infty} |\psi|^2 ds = +\infty.$$

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FLOW CONTROL WITH NONCIRCULAR JETS[1]

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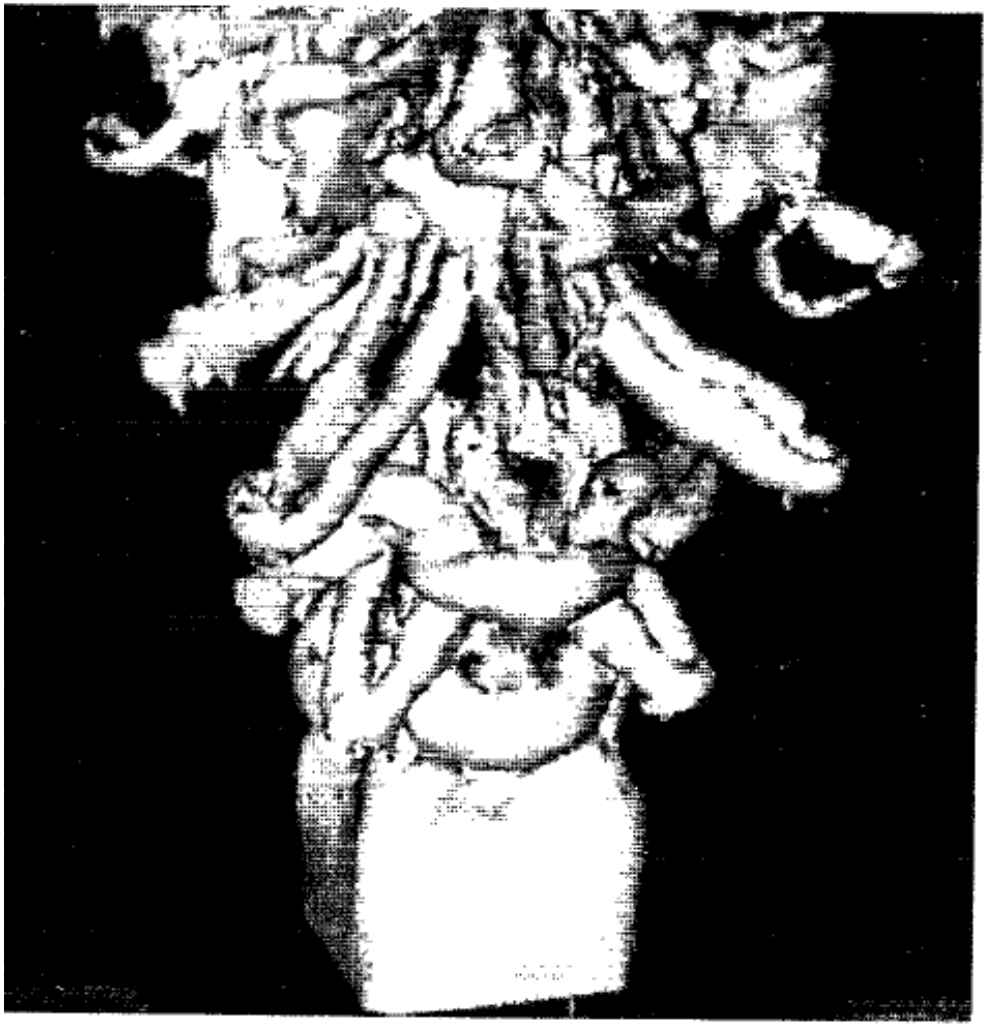
KEY WORDS: vortices, mixing, combustion, entrainment

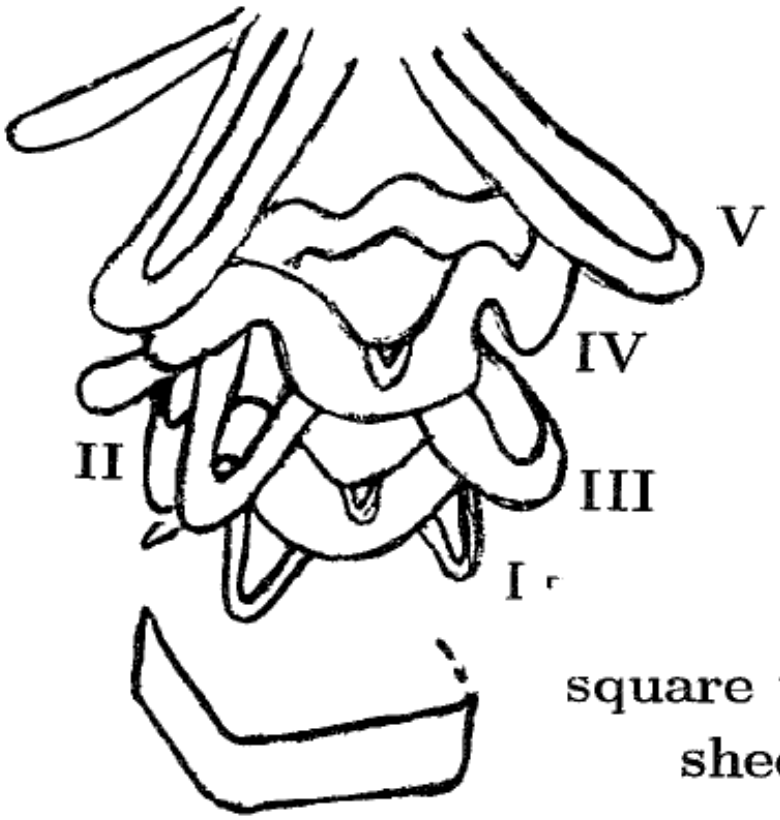
ABSTRACT

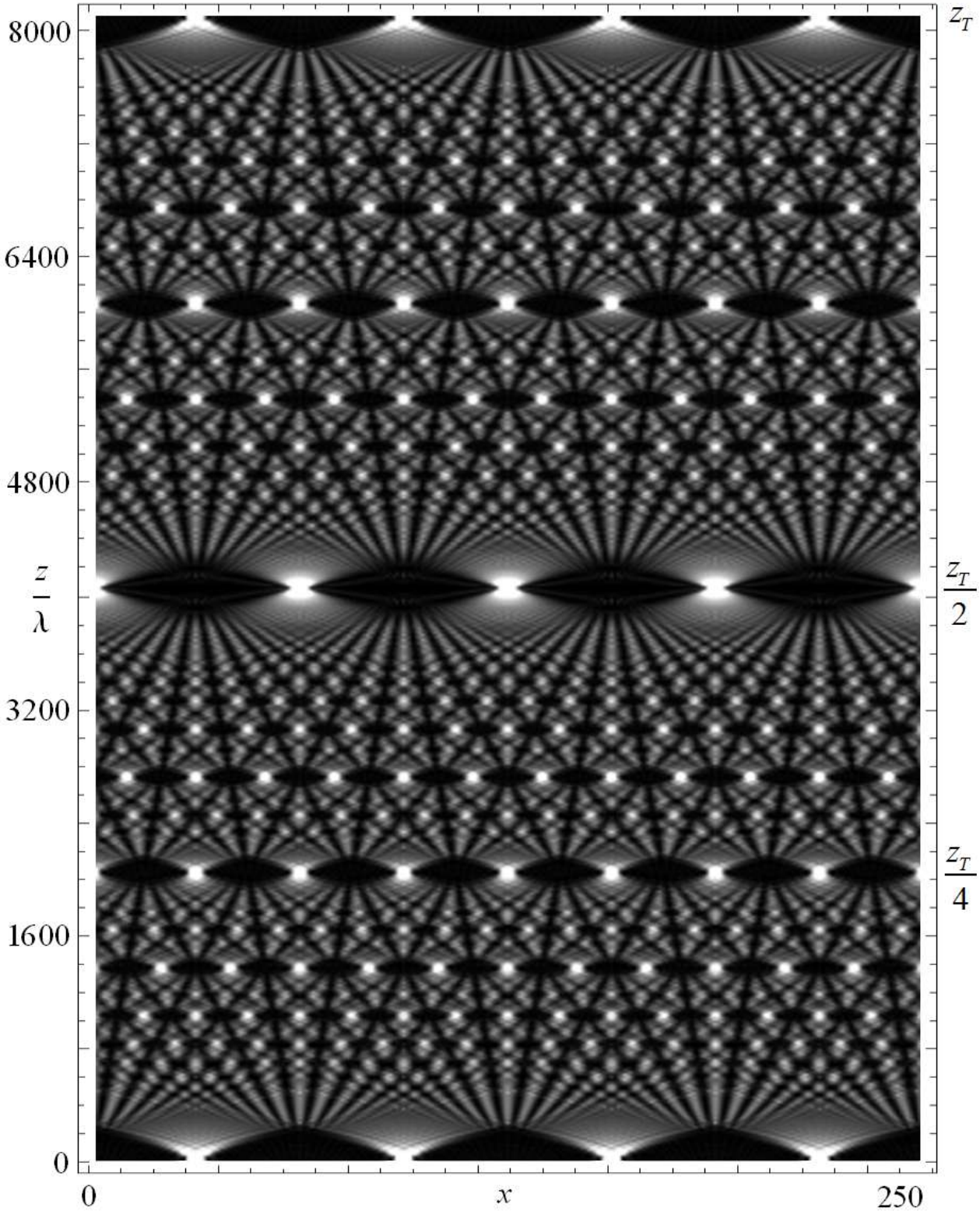
Noncircular jets have been the topic of extensive research in the last fifteen years. These jets were identified as an efficient technique of passive flow control that allows significant improvements of performance in various practical systems at a relatively low cost because noncircular jets rely solely on changes in the geometry of the nozzle. The applications of noncircular jets discussed in this review include improved large- and small-scale mixing in low- and high-speed flows, and enhanced combustor performance, by improving combustion efficiency, reducing combustion instabilities and undesired emissions. Additional applications include noise suppression, heat transfer, and thrust vector control (TVC).

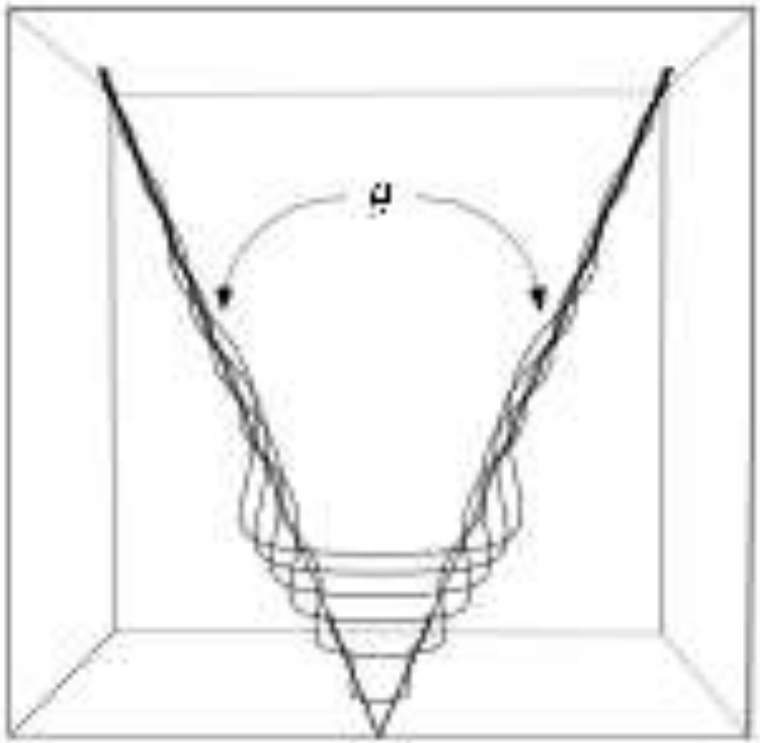
The flow patterns associated with noncircular jets involve mechanisms of vortex evolution and interaction, flow instabilities, and fine-scale turbulence augmentation. Stability theory identified the effects of initial momentum thickness distribution, aspect ratio, and radius of curvature on the initial flow evolution. Experiments revealed complex vortex evolution and interaction related to self-induction and interaction between azimuthal and axial vortices, which lead to axis switching in the mean flow field. Numerical simulations described the details and clarified mechanisms of vorticity dynamics and effects of heat release and reaction on noncircular jet behavior.

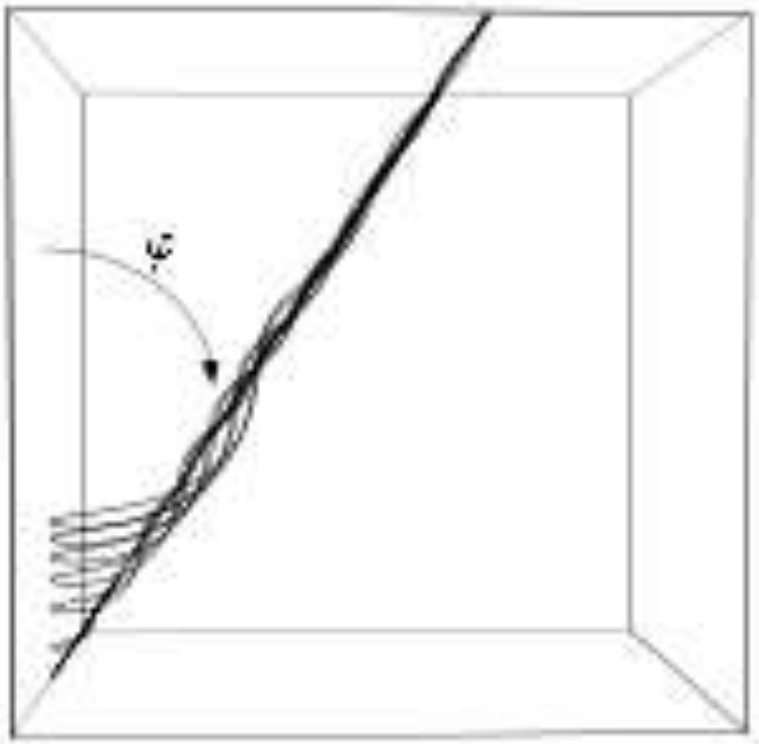


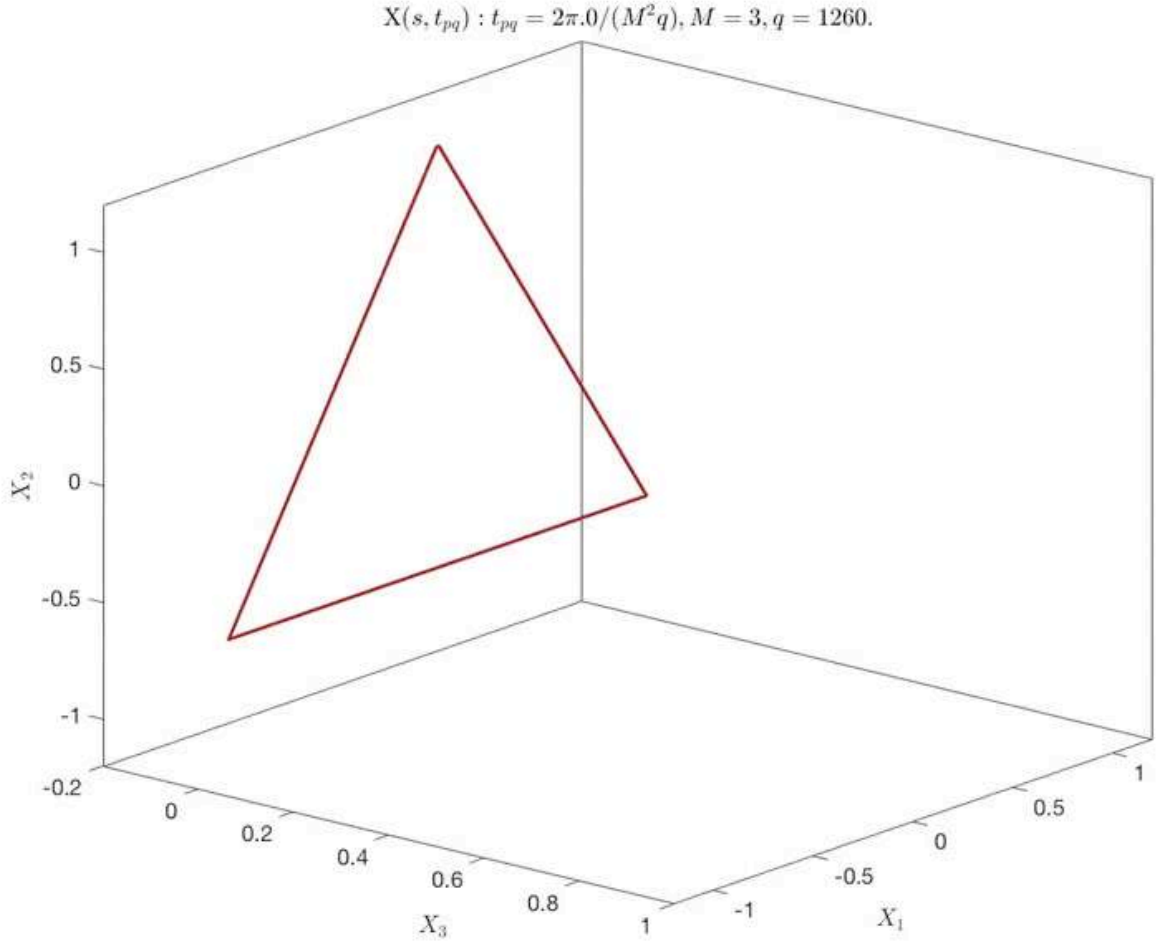
I, III, V : hairpin (braid) vortices
II, IV : deformed vortex rings

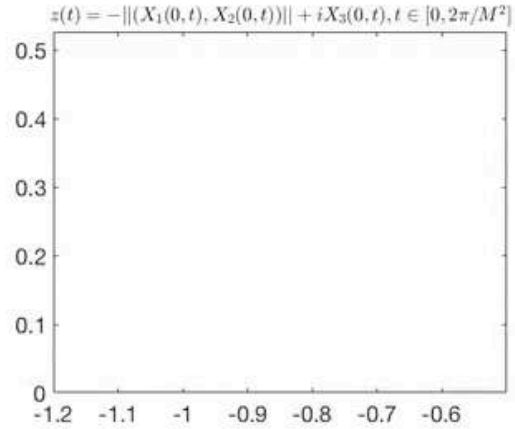


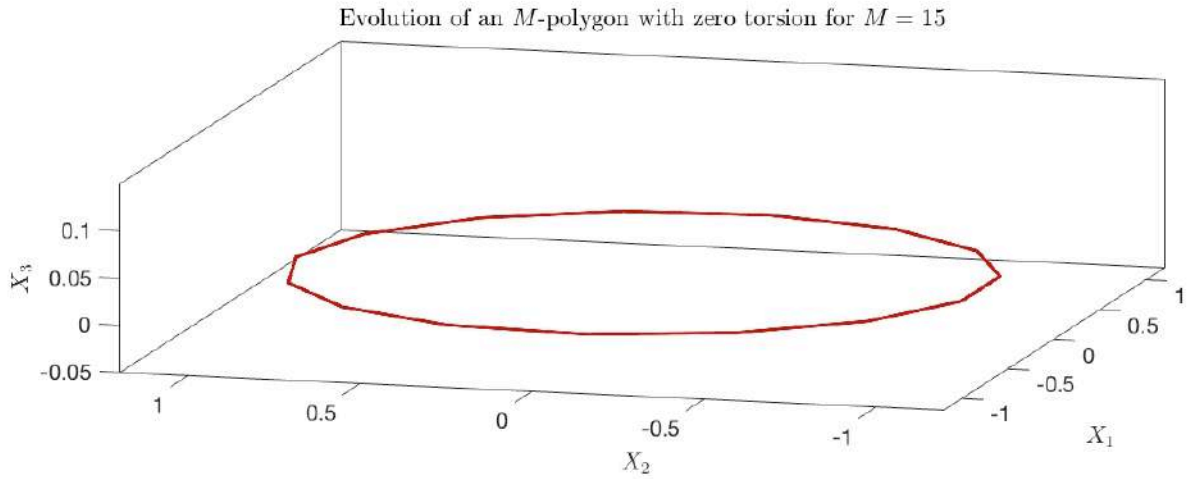

8000
6400
4800
$\frac{z}{\lambda}$
3200
1600
0
0
x
250
z_T
$\frac{z_T}{2}$
$\frac{z_T}{4}$

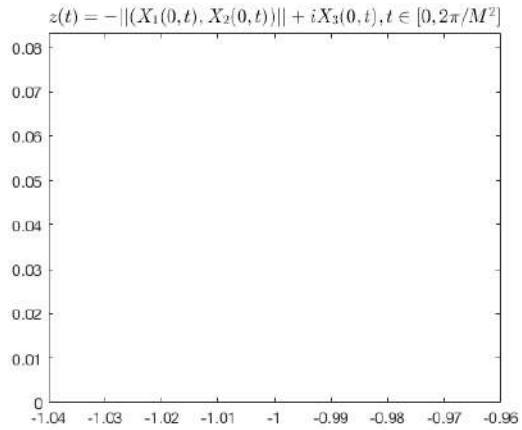


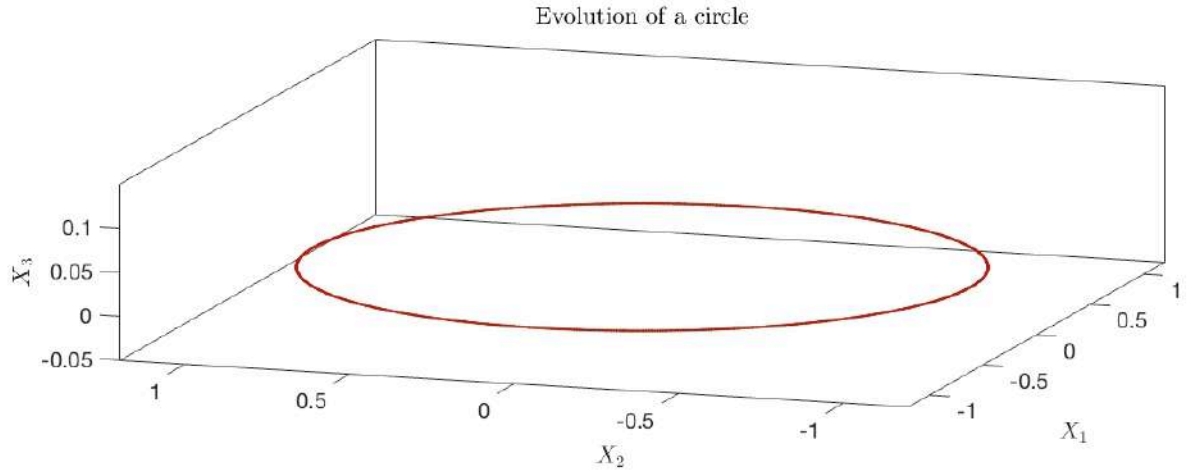


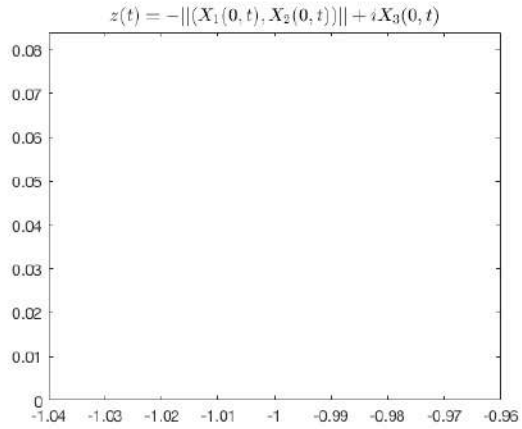

$\mathrm{X}(s, t_{pq}) : t_{pq} = 2\pi.0/(M^2 q), M = 3, q = 1260.$
1
0.5
0
-0.5
-1
X_2
-0.2
0
0.2
0.4
0.6
0.8
1
X_3
1
0.5
0
-0.5
-1
X_1


$z(t) = -||(X_1(0,t), X_2(0,t))|| + iX_3(0,t), t \in [0, 2\pi/M^2]$
0.5
0.4
0.3
0.2
0.1
0
-1.2
-1.1
-1
-0.9
-0.8
-0.7
-0.6


Evolution of an M-polygon with zero torsion for $M = 15$
X_3
0.1
0.05
0
-0.05
1
0.5
0
-0.5
-1
X_2
1
0.5
0
-0.5
-1
X_1


$z(t) = -||(X_1(0,t), X_2(0,t))|| + iX_3(0,t), t \in [0, 2\pi/M^2]$
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-1.04
-1.03
-1.02
-1.01
-1
-0.99
-0.98
-0.97
-0.96


Evolution of a circle
X_3
0.1
0.05
0
-0.05
1
0.5
0
-0.5
-1
X_2
1
0.5
0
-0.5
-1
X_1


$z(t) = -||(X_1(0,t), X_2(0,t))|| + iX_3(0,t)$
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0
-1.04
-1.03
-1.02
-1.01
-1
-0.99
-0.98
-0.97
-0.96

Riemann's non-differentiable function

Integrating the Fourier series in time and evaluating at $x = 0$ we get

$$\phi(t) = i \int_0^t u(0,\tau)\, d\tau = \sum_{k\in\mathbb{Z}} \frac{e^{-4\pi^2 i k^2 t} - 1}{-4\pi^2 k^2},$$

which is essentially Riemann's non-differentiable function.

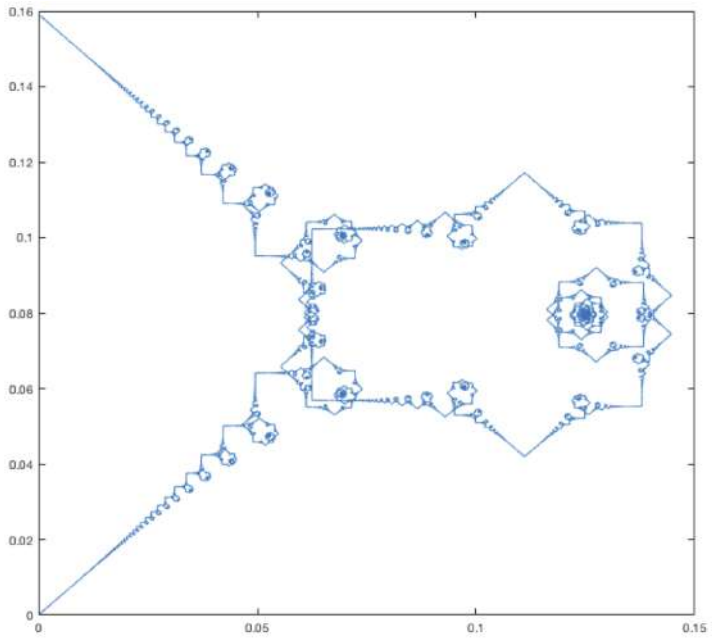
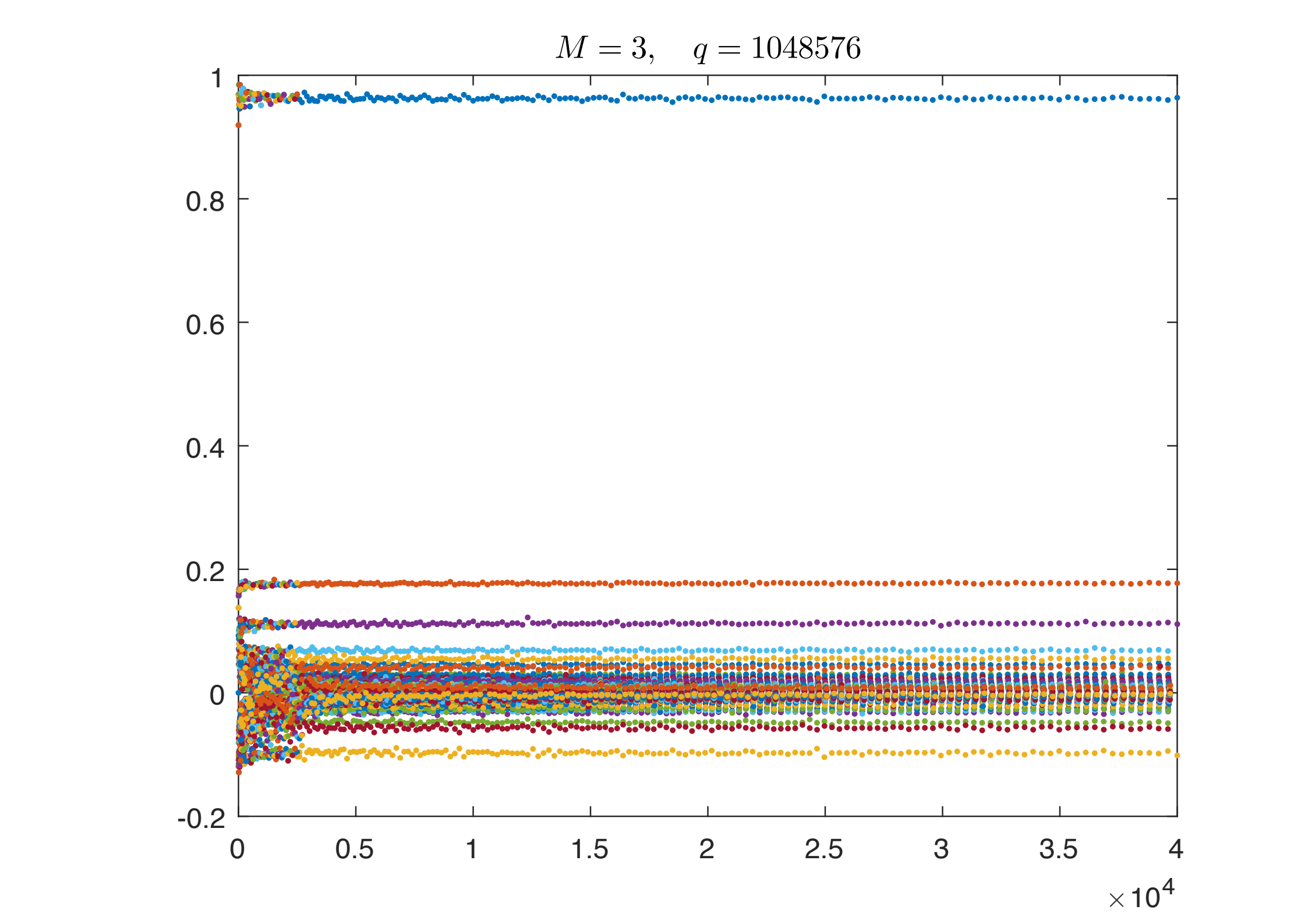
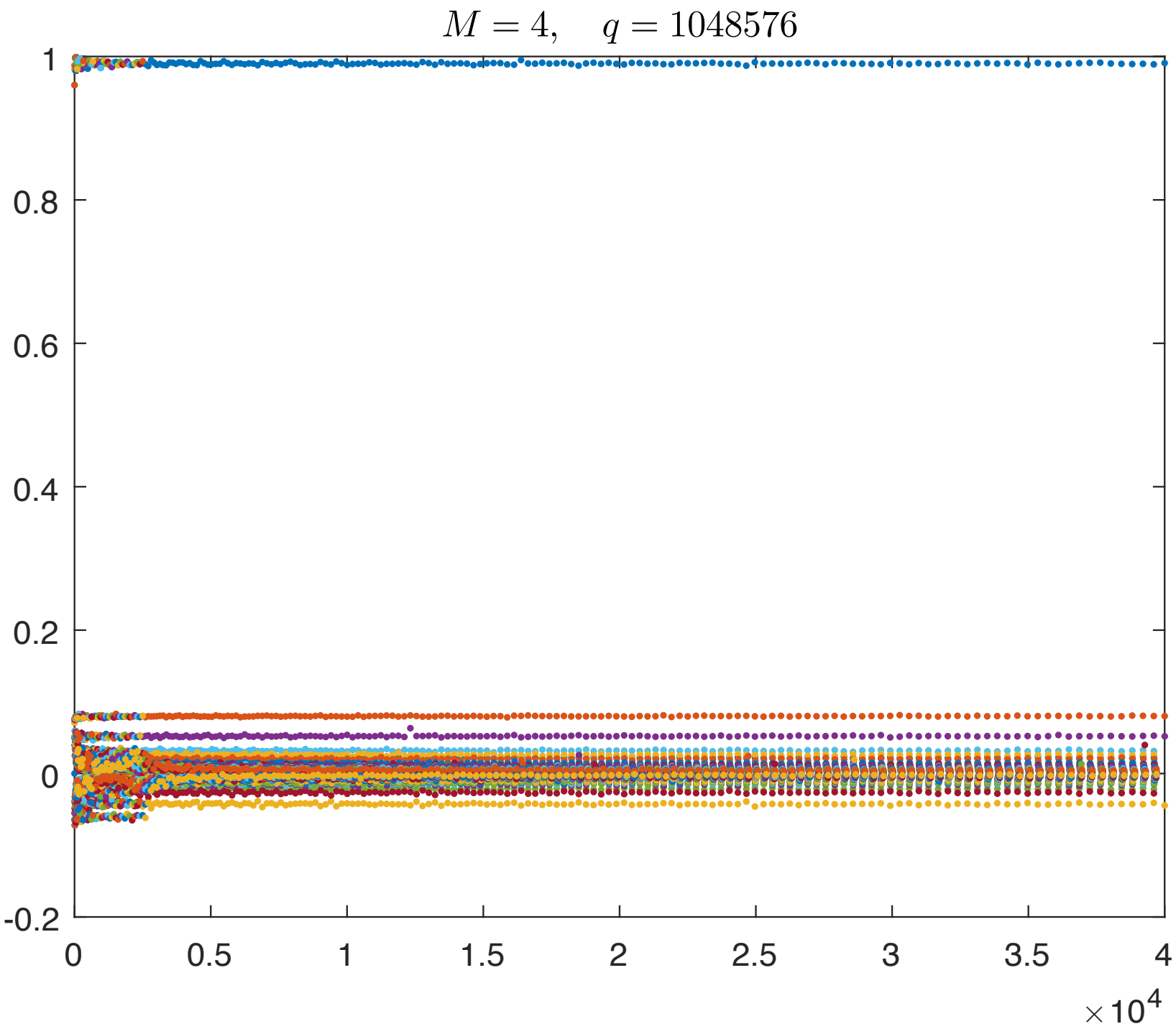


Figure: De la Hoz, Vega: *Vortex filament equation for a regular polygon*, Nonlinearity **27** (2014), 3031-3057


$M = 3, \quad q = 1048576$
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.5
1
1.5
2
2.5
3
3.5
4
$\times 10^4$


$M = 4, \quad q = 1048576$
1
0.8
0.6
0.4
0.2
0
-0.2
0
0.5
1
1.5
2
2.5
3
3.5
4
$\times 10^4$

- Berry and Goldberg, Talbot Effect ’88,
- Duistermaat ’91,
- Oskolkov ’92,
- Jaffard, multifractal ’96,
- Kapitanski, Rodnianski ’99,
- Olver ’10
- Chen-Olver ’12 ’14; Olver-Sheils ’17 Olver-Tsatis ’18
- Jerrard–Smets ’15,
- Erdogan–Tzirakis ’13,
- De la Hoz–Vega ’13, ’17 critical regularity,
- Banica–Vega ’18 critical regularity.

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$$\psi = \alpha + i\beta$$

1–D cubic NLS with several Dirac masses

We consider distributions

$$f = \sum_{k\in\mathbb{Z}} \alpha_k \delta_k .$$

Their Fourier transform on $\mathbb{R}$ writes

$$\hat{f}(\xi) = \sum_{k\in\mathbb{Z}} \alpha_k e^{-ik\xi},$$

and in particular $\hat{f}$ is $2\pi-$periodic. Imposing

$$\|\{\alpha_k\}\|_{l^{2,s}} := \sum_{k\in\mathbb{Z}} (1+|k|)^{2s} |\alpha_k|^2 < \infty$$

translates into $\hat{f} \in H^s(0,2\pi)$.
We define

$$H^s_{pF} := \{f \in \mathcal{S}'(\mathbb{R}),\ \hat{f}(x+2\pi) = \hat{f}(x), \hat{f} \in H^s(0,2\pi)\}.$$

Result for 1–D cubic NLS with several Dirac data

Theorem (Banica-V. 18).– Let $s > \frac{1}{2}$, $0 < \gamma < 1$, $\{\alpha_k\} \in l^{2,s}$ and $M = \sum_{k\in\mathbb{Z}} |\alpha_k|^2$.

We consider the 1-D cubic NLS renormalized equation:

$$i\partial_t u + \Delta u \pm \frac{1}{2}(|u|^2 - \frac{M}{2\pi t})u = 0.$$

There exists $T > 0$ and a unique solution on $(0, T)$ of the form

$$u(t,x) = \sum_{k\in\mathbb{Z}} e^{\mp i \frac{|\alpha_k|^2}{4\pi} \log t} (\alpha_k + R_k(t)) e^{it\Delta}\delta_k(x),$$

with

$$\sup_{0<t<T} t^{-\gamma} \|\{R_k(t)\}\|_{l^{2,s}} + t\,\|\{\partial_t R_k(t)\}\|_{l^{2,s}} < C.$$

Moreover, if $s \geq 1$ then the solution can be extended to $(0, \infty)$.

Result for 1–D cubic NLS with several Dirac data

Theorem $\star$ **(Banica-V. 18).**– Let $a \in \mathbb{R}$, $N \in \mathbb{N}$, $M = \left(N - \frac{1}{2}\right) a^2$ and $s > \frac{1}{2}$.

We consider as initial data a finite sum of N Dirac masses with coefficients $\alpha_k \in \mathbb{C}$ of equal modulus $|\alpha_k| = a$:

$$\begin{cases} i\partial_t u + \Delta u \pm \frac{1}{2}\left(|u|^2 - \frac{M}{2\pi t}\right) u = 0, \\ u(0) = \sum_k \alpha_k \delta_k. \end{cases}$$

There exists a unique global solution:

$$u(t) = e^{it\Delta} u(0) \pm i e^{it\Delta} \int_0^t e^{-i\tau\Delta} \left(\left(|u(\tau)|^2 - \frac{M}{2\tau}\right) u(\tau)\right) \frac{d\tau}{2},$$

such that $\widehat{e^{-it\Delta} u(t)} \in \mathcal{C}^1(\mathbb{R}, H^s(0, 2\pi))$ with

$$\|e^{-it\Delta} u(t) - u(0)\|_{H^s_{pF}} \leq C t^\gamma, \quad \forall t \in (-1, 1).$$

Evolution of polygonal lines by the binormal flow

Theorem (Banica-V. 18).– Let $\chi_0(x)$ be a polygonal line parametrized by arc length with corners located at $x = k \in \mathbb{Z}$, of angles θ_k s.t. $\{a_k\}$ defined by $\sin(\frac{\theta_k}{2}) = e^{-\pi \frac{a_k^2}{2}}$ belongs to $l^{2,3}$. Then there exists $\chi(t)$ smooth solution of the binormal flow on $\mathbb{R}^*$,

solution in the weak sense on $\mathbb{R}$ with

$$|\chi(t,x) - \chi_0(x)| \leq C\sqrt{t}, \quad \forall x \in \mathbb{R}, |t| \leq 1.$$

Energy transfer

Theorem.– For $t > 0$ we have the following conservation law:

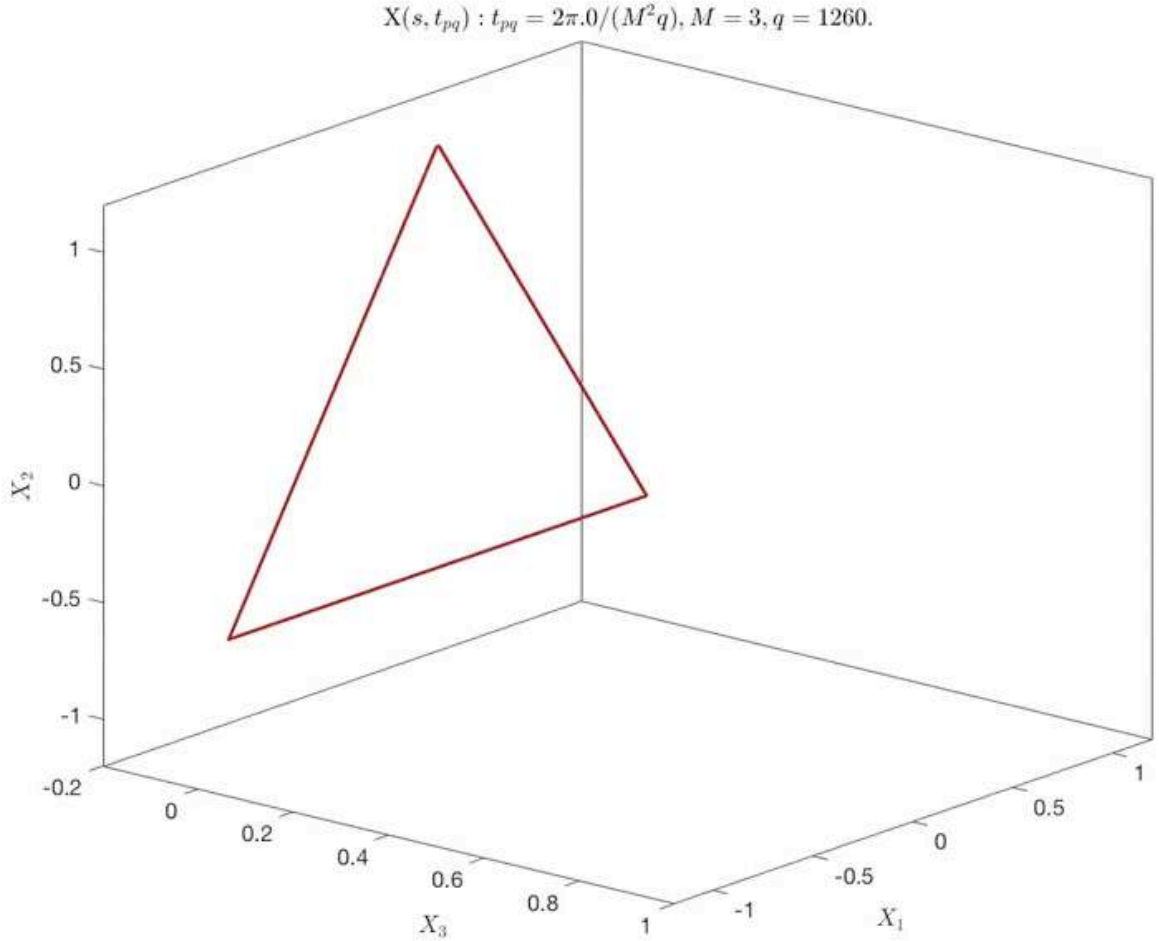
$$\|T(t)\|^2_{L^2_{sc}} = 4\pi \sum_j |\alpha_j|^2,$$

but

$$(*) \qquad \|T(0)\|^2_{L^2_{sc}} = 4 \sum_j (1 - e^{-\pi |\alpha_j|^2}),$$

where

$$\|T(t)\|^2_{L^2_{sc}} := \lim_{k \to \infty} \int_k^{k+1} |\widehat{T_x}(t, \xi)|^2 d\xi.$$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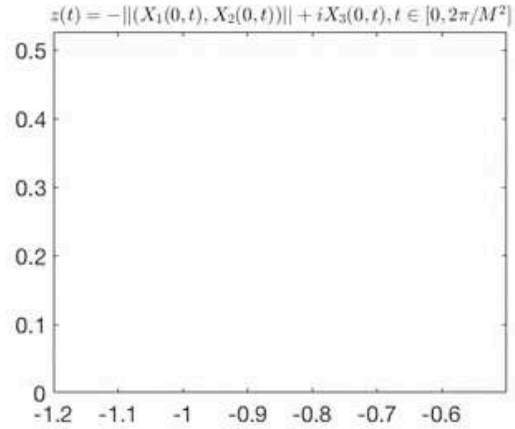
0
-0.5
-1
X_1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0.1
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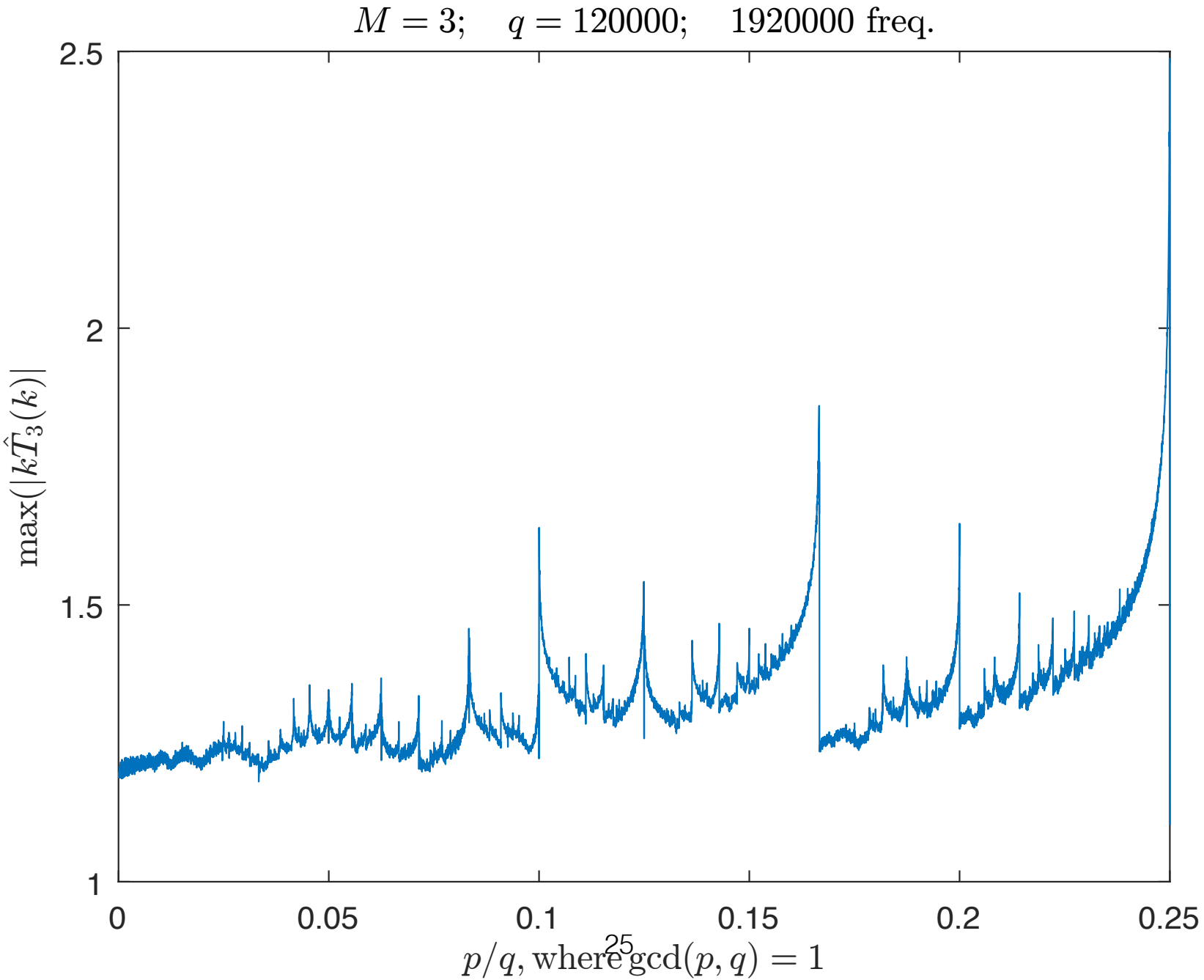
-1.2
-1.1
-1
-0.9
-0.8
-0.7
-0.6

Energy Transfer, NLTe



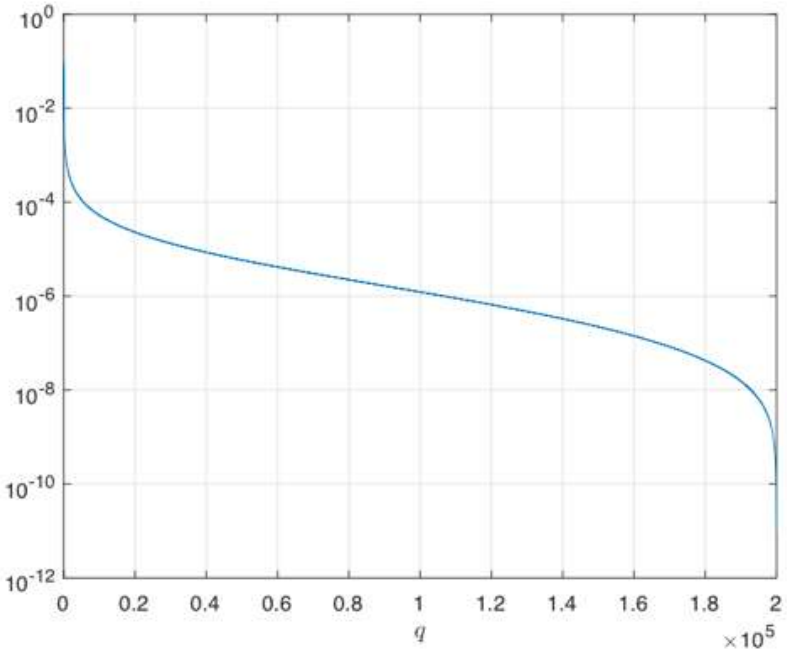


Figure 10: $|\sqrt{2}\max_{t_{pq}} \|\widehat{T_{1,s}}(t_{pq})\|_\infty - a\ln(q) - b|$, for $a = 0.258039752572419$, $b = 0.152992510344641$.

THANK YOU FOR YOUR ATTENTION