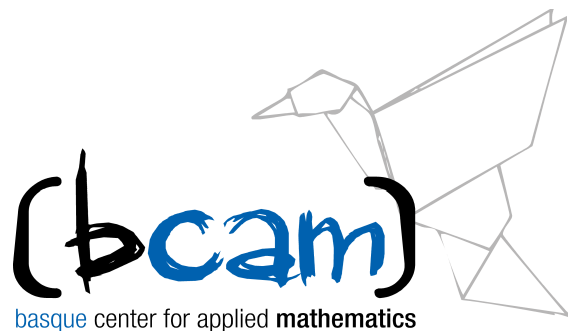


Riemann's non-differentiable and the binormal flow

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Summary

(1) The analytical problem:

“Continuous functions nowhere differentiable”

(2) The geometric problem:

“Evolution of vortex filaments (smoke rings)”

$$\chi_t = cb$$

How does a corner evolve? And several corners?

(3) The PDE problem:

- 1d-Cubic NLS: $u_0 = \sum_{j \in \mathbb{Z}} a_j \delta(x - j)$
- Schrödinger map

(4) Is there a connection with turbulence?

Conjecture: Yes (Non-circular jets)

Multifractal formalism (Frisch-Parisi)

Intermittency

Riemann's function

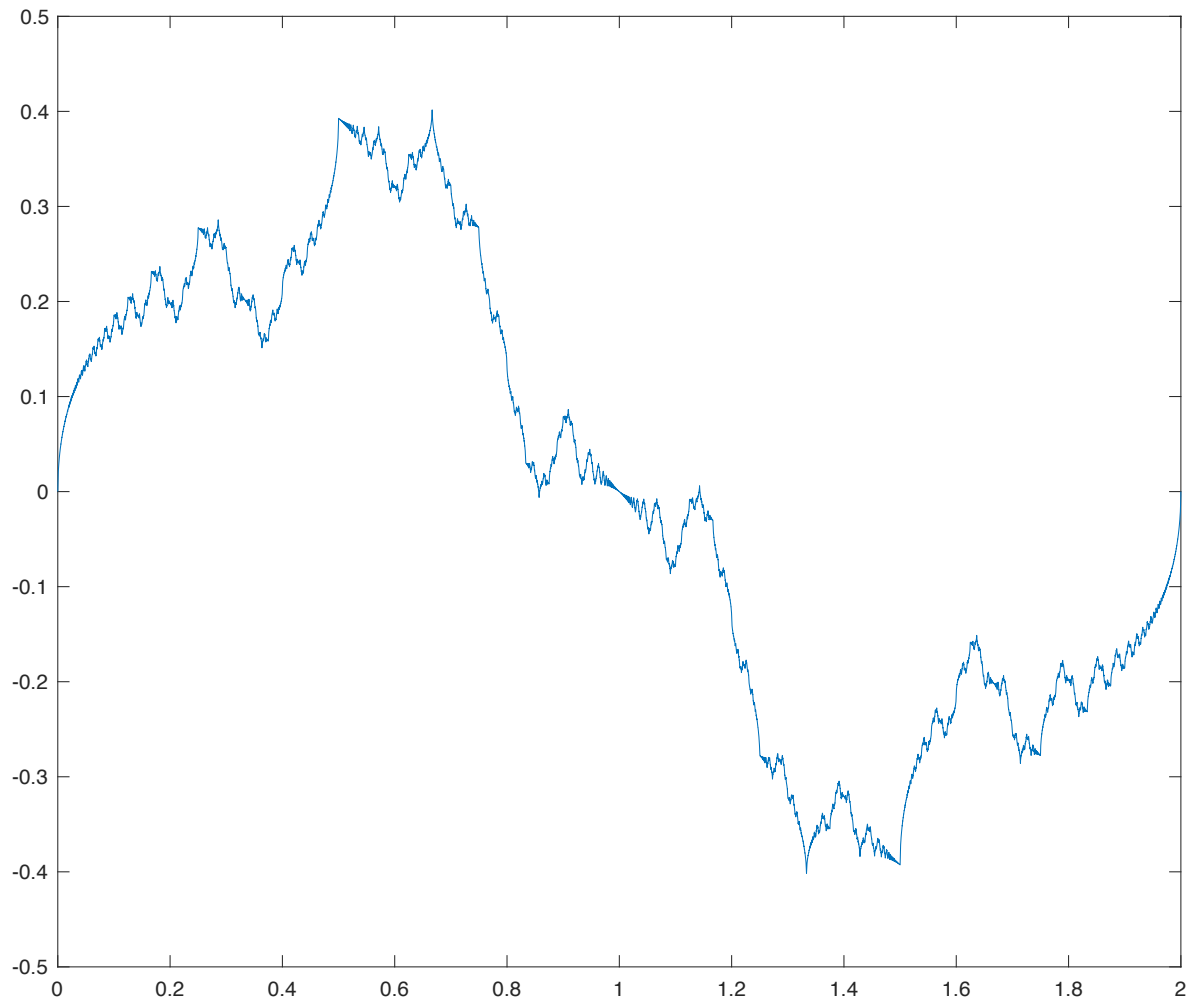
$$\varphi_R(t) = \sum_{j=1}^{\infty} \frac{\sin(tj^2)}{j^2} \quad (\sim 1860)$$

- Hardy 1915 (H–Littlewood circle method)
- Gerver 1960 (Riemann was wrong)

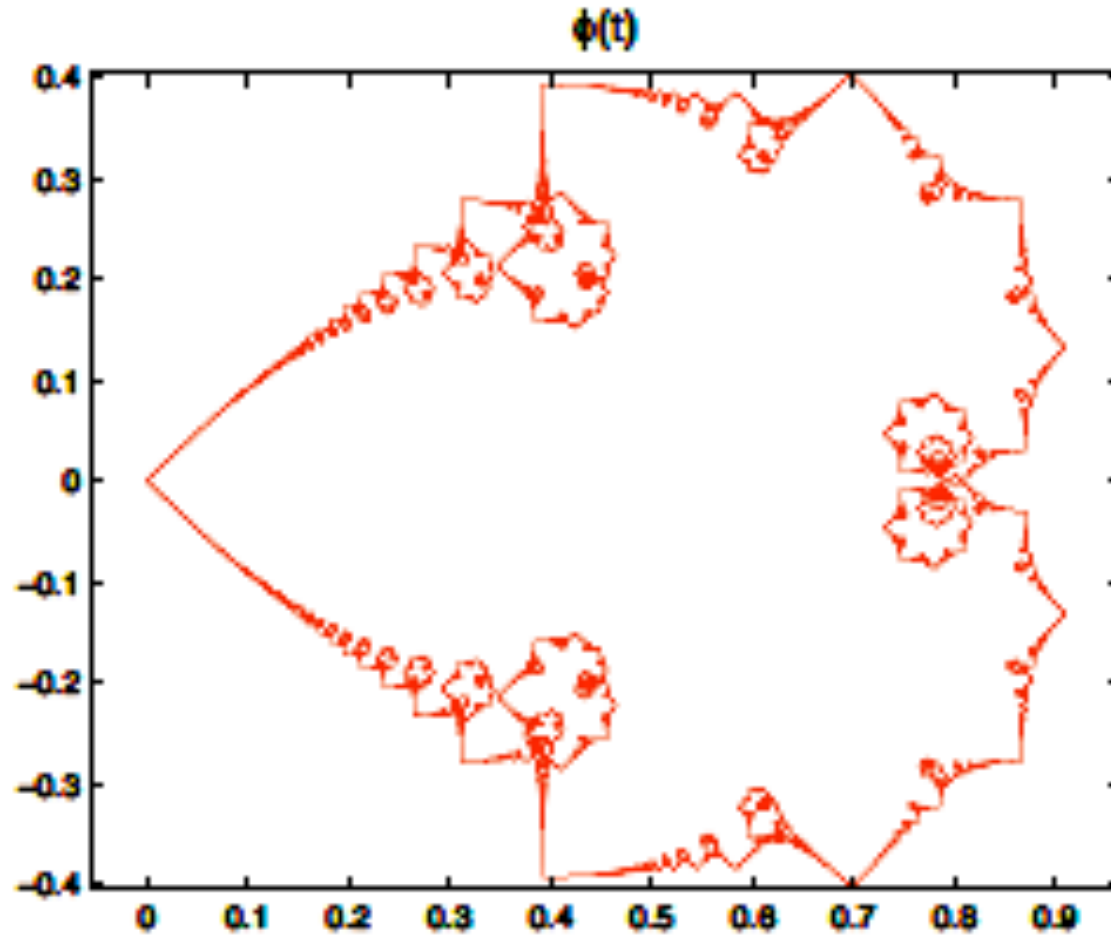
At $t_{p,q} = \pi p/q$ p, q odd, the derivative exists and is $-1/2$

$$\varphi_D(t) = \sum_{j=1}^{\infty} \frac{e^{itj^2}}{ij^2} \quad \text{Duistermaat 1991}$$

Fractal behavior of the graph.



$$\phi(t) = \sum_{k \neq 0} \frac{e^{\pi i k^2 t}}{i \pi k^2}, \quad t \in [0, 2]$$



- Jaffard
- Multifractal (Frisch–Parisi conjecture)

Multifractal formalism

Spectrum of singularities (of a function f):

$$d_f(\beta) = H\text{-dim } E_\beta$$

$$E_\beta = \{t_0 \in [0, 2] : f \text{ is } \beta\text{-H\"older at } t_0\}$$

i. e. $\sup \{\alpha : f \in \mathcal{C}^\alpha(t_0)\}$

$$|f(t) - P(t - t_0)| < C|t - t_0|^\alpha$$

Example: Weierstrass functions

$$\mathcal{W}_{a,b}(t) = \sum_{n \neq 0} a^n \cos(b^n t) \quad a < 1 < ab$$

- Nowhere differentiable
- $\alpha = -\lg a / \lg(b)$ (monofractal)

- Antonia, Hopfinger, Gagne, and Anselment experiment 1984
- Frisch–Parisi: multifractal model
- Multifractal formalism (Frisch–Parisi conjecture)

$$d_f(\beta) = \inf_p (\beta p - \eta_f(p) + 1)$$

$$\eta_f(p) = \sup \{s : f \in \beta_p^{s/p, \infty}\}$$

- Jaffard: (1996)

$$d_{\varphi_R}(\beta) = 4\beta - 2 \quad \frac{1}{2} \leq \beta \leq \frac{3}{4}$$

Multifractal formalism is true for φ_R

- $\sum_k \frac{\sin t(ck + d)^2}{(ck + d)^2} \quad c, d \in \mathbb{Z}$

(2) The binormal curvature flow

(BF) • $\chi_t = \chi_s \wedge \chi_{ss} = cb$ c : curvature b : binormal

(SM) • $\chi_s = T$ Schrödinger map $T_t = T \wedge T_{ss}$

$$T_s = cn$$

$$n_s = -cT + \tau b$$

$$b_s = -\tau n$$

Examples:

(i) Straight lines.

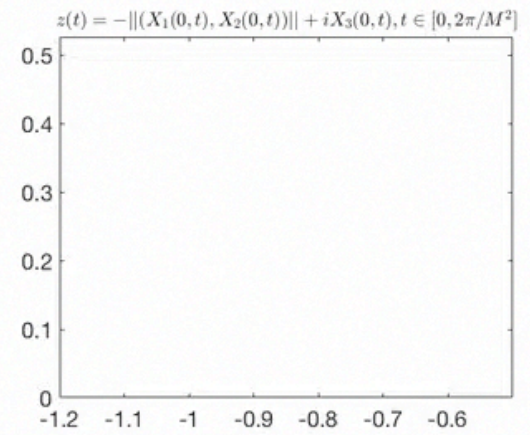
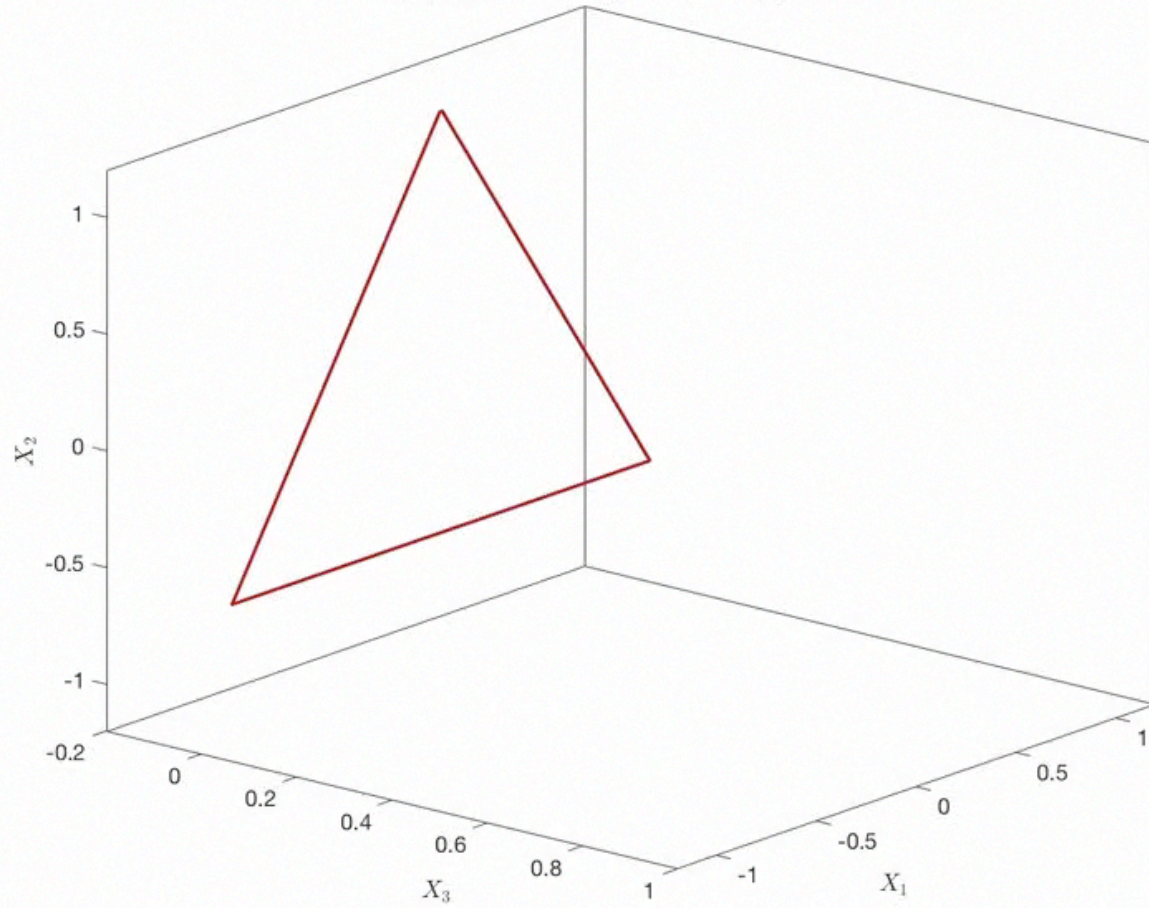
(ii) Circles.

(iii) Helices.

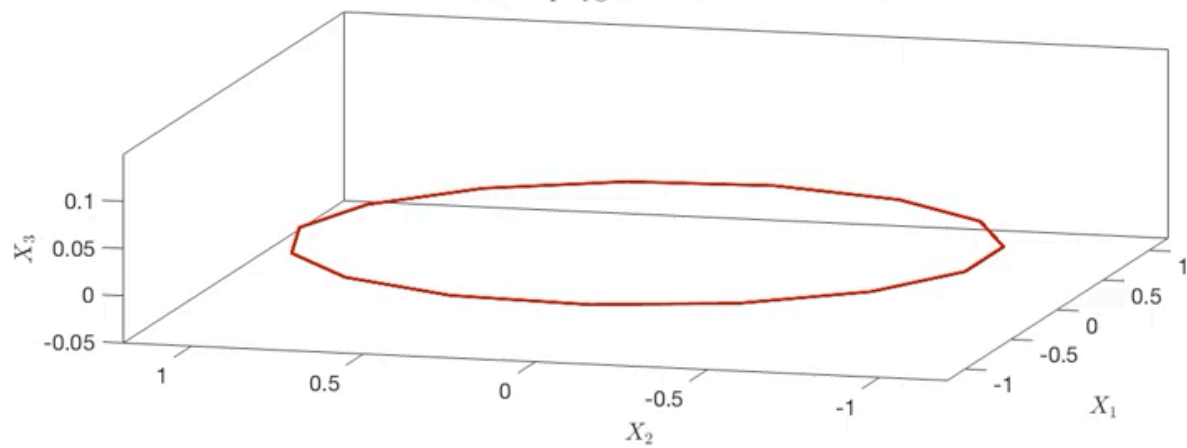




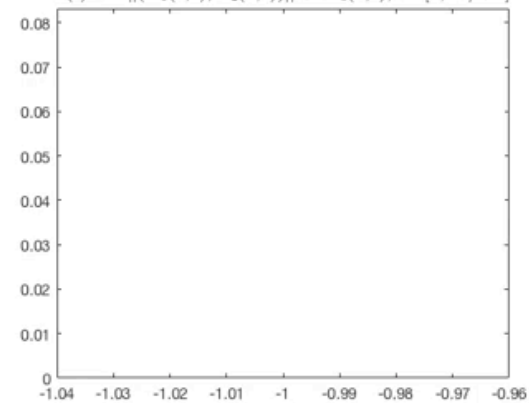
$X(s, t_{pq}) : t_{pq} = 2\pi.0/(M^2q), M = 3, q = 1260.$



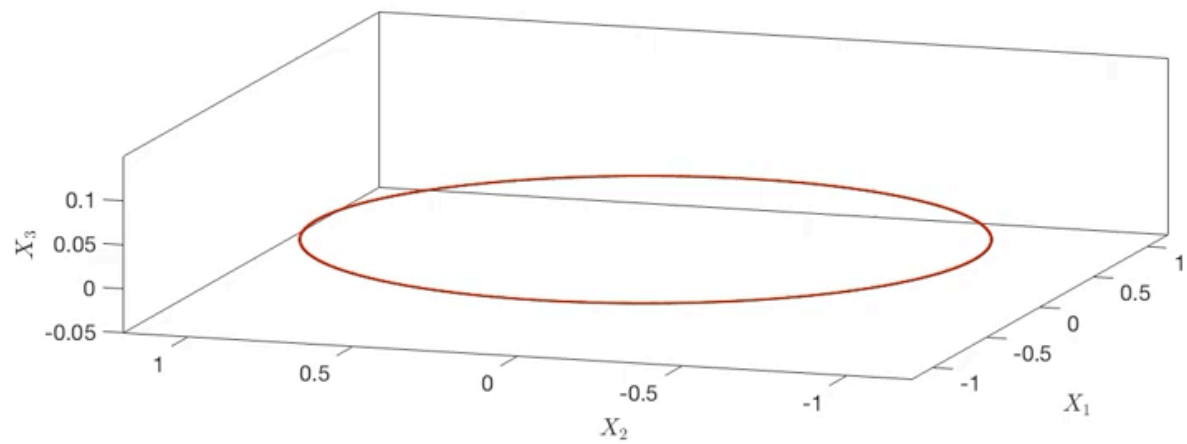
Evolution of an M -polygon with zero torsion for $M = 15$



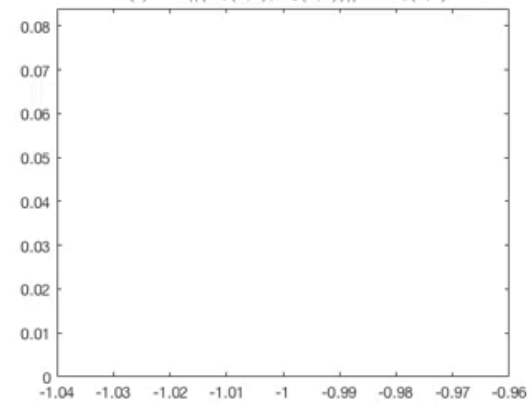
$$z(t) = -\|(X_1(0, t), X_2(0, t))\| + iX_3(0, t), t \in [0, 2\pi/M^2]$$

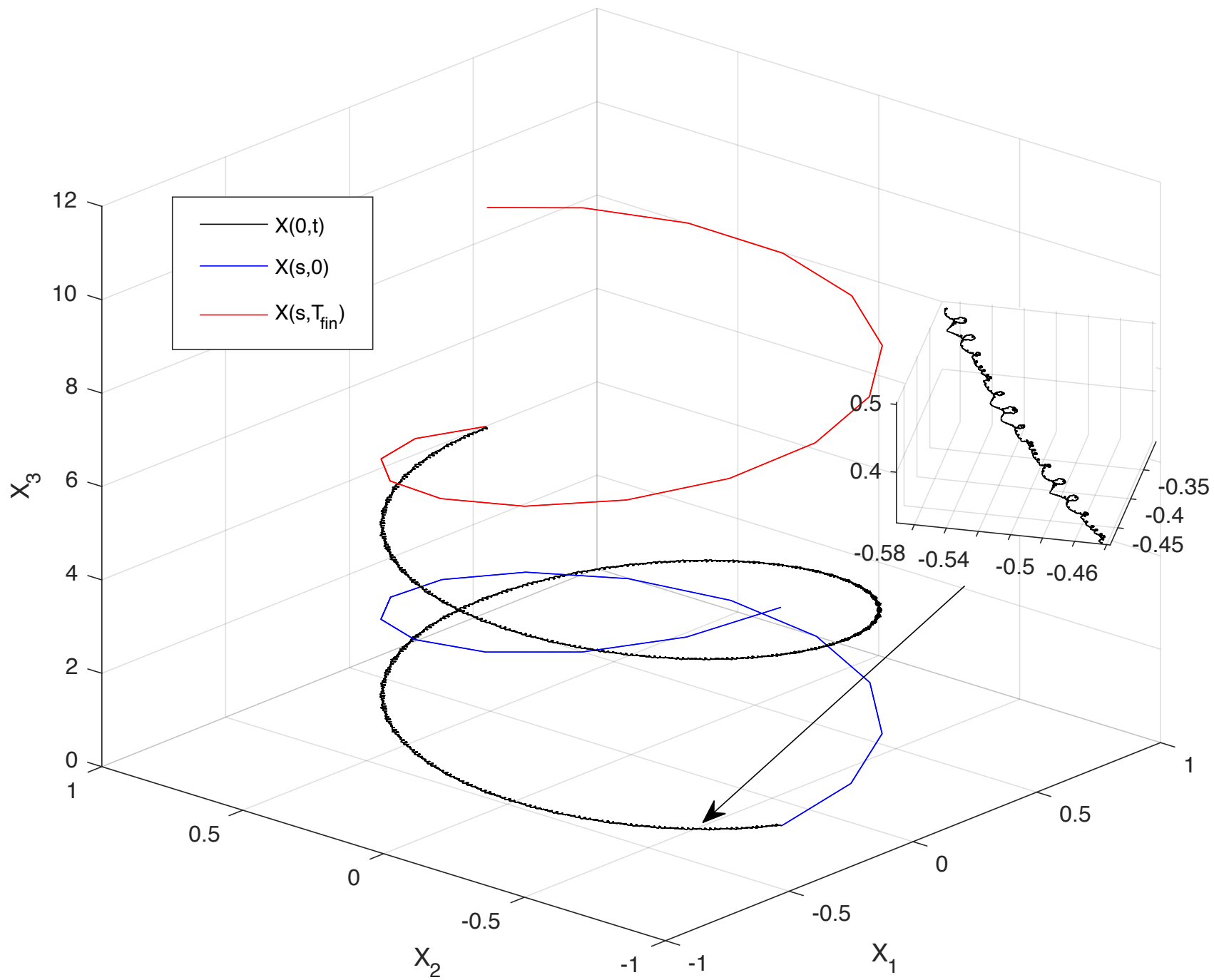


Evolution of a circle



$$z(t) = -\|(X_1(0, t), X_2(0, t))\| + iX_3(0, t)$$





- Buttke 1988
- Peskin, Mcqueen 1994
- Jerrard, Smets 2011
- De La Hoz, Vega 2013, 2018
- Kumar, De La Hoz, Vega 2020

The PDEs

(BF) • $\chi_t = \chi_s \wedge \chi_{ss} = cb$ c : curvature b : binormal

(SM) • $\chi_s = T$ Schrödinger map $T_t = T \wedge T_{ss}$

(NLS) • ψ Hasimoto wave function 1d **NLS** (cubic focusing)

$\chi(0, s)$: skew polygonal line

$T(0, s)$: sequence of points T_j such that $\lim_{j \rightarrow \pm\infty} T_j = A^\pm$

$$\psi(0, s) = \sum a_j \delta(s - j) \quad \sum_j |j|^{1+} |a_j|^2 < +\infty$$

$$\begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}_s = \begin{pmatrix} 0 & \alpha & \beta \\ -\alpha & 0 & 0 \\ -\beta & 0 & 0 \end{pmatrix} \begin{pmatrix} T \\ e_1 \\ e_2 \end{pmatrix}$$

$$\psi = \alpha + i\beta$$

Schrödinger equation

Hasimoto transformation:

$$\psi(s, t) = c(s, t) e^{i \int_0^s \tau(s', t) ds'}$$

$$c = c(s, t) \quad \text{curvature}$$

$$\tau = \tau(s, t) \quad \text{torsion}$$

$$\partial_t \psi(s, t) = i \left(\partial_s^2 \psi \pm \frac{1}{2} (|\psi|^2 + A(t)) \psi \right) \quad A(t) \in \mathbb{R}$$

$$\int_{-\infty}^{\infty} |\psi(s, t)|^2 ds = \int_{-\infty}^{\infty} |\psi(s, 0)|^2 ds = \int_{-\infty}^{\infty} c^2(s, 0) ds$$

In our case

$$\psi(s, t) = \frac{a}{\sqrt{t}} e^{i \frac{s^2}{4t}}, \quad \int_{-\infty}^{\infty} |\psi|^2 ds = +\infty.$$

Selfsimilar solutions: The ODE

$$\chi_t = \chi_s \wedge \chi_{ss} = cb$$

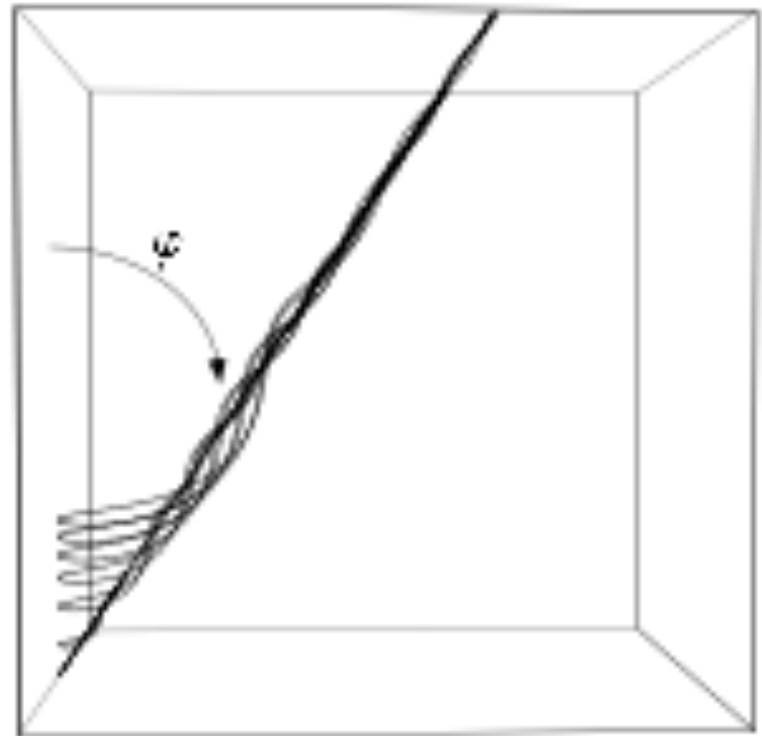
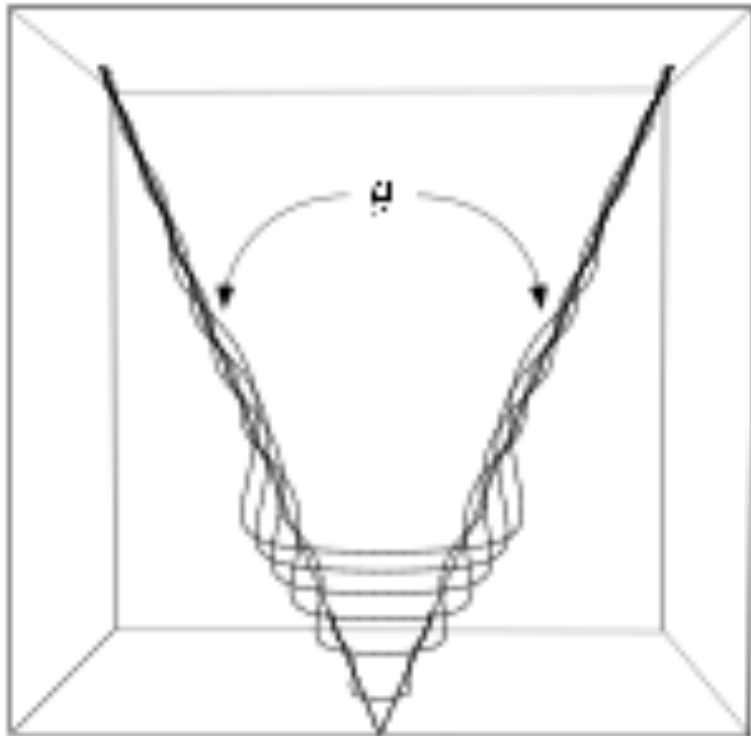
- $\chi(s, t) = \sqrt{t} G(s/\sqrt{t})$
- $\frac{1}{2}G - \frac{s}{2}G' = G' \wedge G''$
- The curvature is constant $c(s) = a_0$
- $G(0) = 2a_0 b(0)$ (trajectory of the corner)
- Given the angle of the corner θ_0 , **what is a_0 ?**

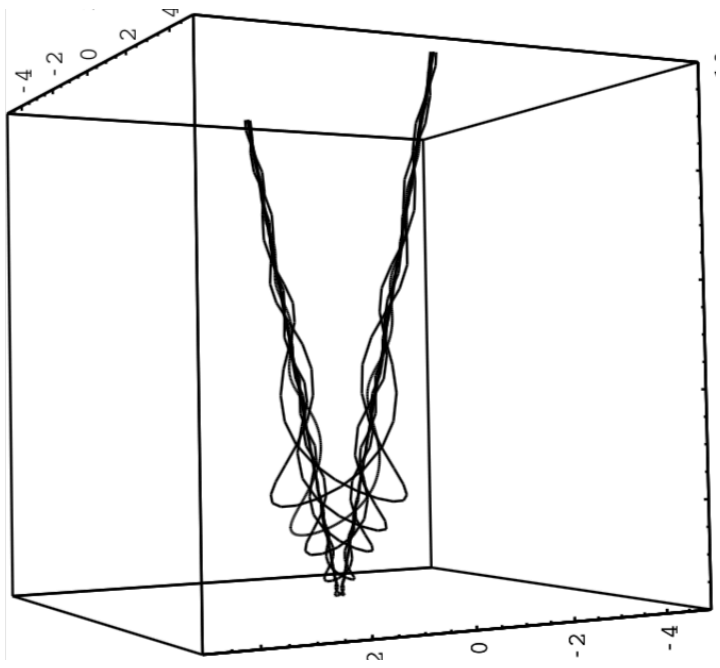
$$\sin \frac{\theta}{2} = e^{-\pi \frac{a_0^2}{2}} \quad (\text{Gutiérrez, Rivas, V. 2003})$$

- New approach with De la Hoz, 2018

If $Z(\eta) = \eta \hat{G}(\sqrt{\eta})$ then

$$Z'' + \left(1 - \frac{a_0^2}{\eta}\right) Z = 0 \quad \eta > 0.$$





1-D cubic NLS with several Dirac masses

We consider distributions

$$f = \sum_{k \in \mathbb{Z}} \alpha_k \delta_k, \quad \delta_k(\cdot) = \delta(\cdot - k)$$

Their Fourier transform on \mathbb{R} writes

$$\hat{f}(\xi) = \sum_{k \in \mathbb{Z}} \alpha_k e^{-ik\xi},$$

and in particular \hat{f} is 2π -periodic. Imposing

$$\|\{\alpha_k\}\|_{l^{2,s}} := \sum_{k \in \mathbb{Z}} (1 + |k|)^{2s} |\alpha_k|^2 < \infty$$

translates into $\hat{f} \in H^s(0, 2\pi)$.

We define

$$H_{pF}^s := \{f \in \mathcal{S}'(\mathbb{R}), \hat{f}(\xi + 2\pi) = \hat{f}(\xi), \hat{f} \in H^s(0, 2\pi)\}.$$

Result for 1-D cubic NLS with several Dirac data

Theorem (Banica-V. 18).— Let $s > \frac{1}{2}$, $0 < \gamma < 1$, $\{\alpha_k\} \in l^{2,s}$ and $M = \sum_{k \in \mathbb{Z}} |\alpha_k|^2$.

We consider the 1-D cubic NLS renormalized equation:

$$i\partial_t u + \Delta u \pm \frac{1}{2}(|u|^2 - \frac{M}{2\pi t})u = 0.$$

There exists $T > 0$ and a unique solution on $(0, T)$ of the form

$$u(t, x) = \sum_{k \in \mathbb{Z}} e^{\mp i \frac{|\alpha_k|^2}{4\pi} \log t} (\alpha_k + R_k(t)) e^{it\Delta} \delta_k(x),$$

with

$$\sup_{0 < t < T} t^{-\gamma} \|\{R_k(t)\}\|_{l^{2,s}} + t \|\{\partial_t R_k(t)\}\|_{l^{2,s}} < C.$$

Moreover, if $s \geq 1$ then the solution can be extended to $(0, \infty)$.

Evolution of polygonal lines by the binormal flow

Theorem (Banica-V. 18).— Let $\chi_0(x)$ be a polygonal line parametrized by arc length with corners located at $x = k \in \mathbb{Z}$, of angles θ_k s.t. $\{a_k\}$ defined by $\sin(\frac{\theta_k}{2}) = e^{-\pi \frac{a_k^2}{2}}$ belongs to $l^{2,3}$. Then there exists $\chi(t)$ smooth solution of the binormal flow on \mathbb{R}^* ,

solution in the weak sense on \mathbb{R} with

$$|\chi(t, x) - \chi_0(x)| \leq C\sqrt{|t|}, \quad \forall x \in \mathbb{R}, |t| \leq 1.$$

Refined analysis for some families of polygonal lines

Let $n \in \mathbb{N}^*$, $\nu \in]0, 1]$, $\Theta > 0$.

From now on we focus on particular classes of initial data: polygonal lines $\chi_n(0)$ with finite but many corners located at $j \in \mathbb{Z}$ with $|j| \leq n^\nu$, of same torsion ω_0 and angles θ_n such that

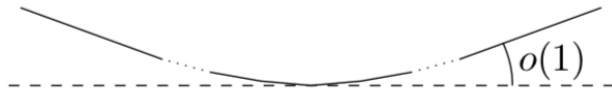
$$\lim_{n \rightarrow \infty} n(\pi - \theta_n) = \Theta,$$

and we suppose without loss of generality

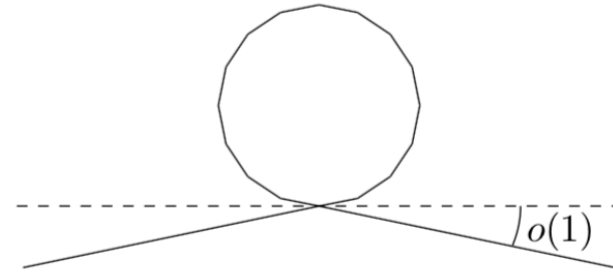
$$\chi_n(0, 0) = (0, 0, 0), \quad \partial_x \chi_n(0, 0^\pm) = \left(\sin \frac{\theta_n}{2}, \pm \cos \frac{\theta_n}{2}, 0 \right).$$

- In the NLS fixed point argument we use l^1 -spaces, take advantage of the sizes of nonresonant phases, get better estimates and a time of existence $T > 0$ independent of n .

The initial data are the polygonal lines:



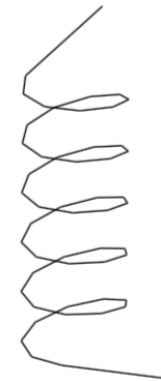
$\chi_n(0)$ planar approximation of a line



$\chi_n(0)$ planar approximation of a (multi-)loop



$\chi_n(0)$ non-planar approximation of a line



$\chi_n(0)$ approximation of multi-turns of helices

Theorem (a Frish-Parisi multifractal behaviour)

For the previous solutions with torsion $\omega_0 \in \pi\mathbb{Q}$ we have the following description of the trajectory of the corner $\chi_n(t, 0)$, uniformly on $(0, T)$:

$$n \chi_n(t, 0) - (0, \Re(\tilde{\mathfrak{R}}(t)), \Im(\tilde{\mathfrak{R}}(t))) \xrightarrow{n \rightarrow \infty} 0.$$

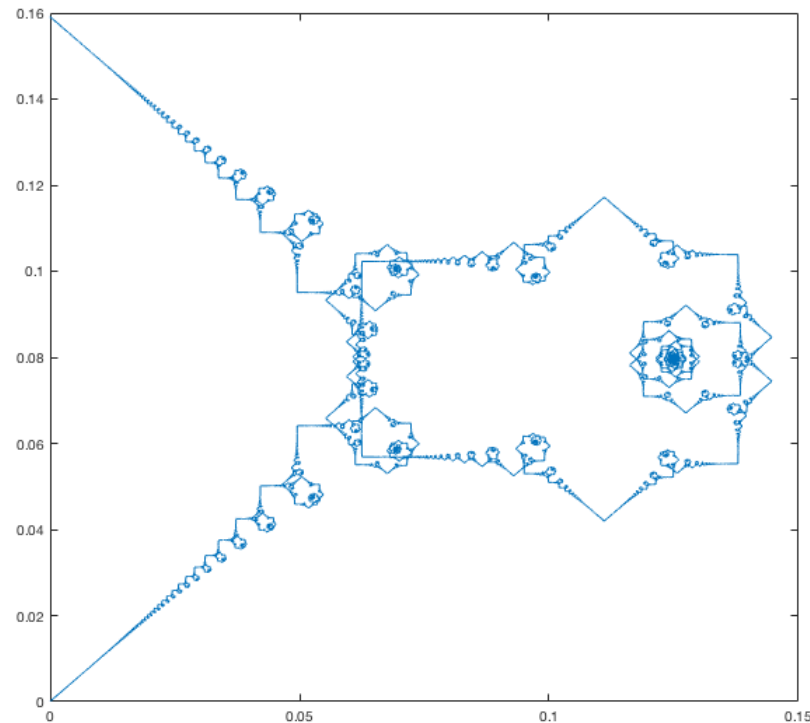
The function $\tilde{\mathfrak{R}}$ is multifractal, and its spectrum of singularities $d_{\tilde{\mathfrak{R}}}$ satisfies the multifractal formalism of Frisch-Parisi:

$$d_{\tilde{\mathfrak{R}}}(\beta) := \dim_{\mathcal{H}} \{t, \tilde{\mathfrak{R}} \in \mathcal{C}^\beta(t)\} = \inf_p (\beta p - \eta_{\tilde{\mathfrak{R}}}(p) + 1),$$
$$\eta_{\tilde{\mathfrak{R}}}(p) := \sup \{s, \tilde{\mathfrak{R}} \in B_p^{\frac{s}{p}, \infty}\},$$

a model for predicting the structure function exponents in turbulent flows.

In the torsion-free case $\tilde{\mathfrak{R}}(t) = -\Theta \frac{\Re(4\pi^2 t)}{4\pi^2}$, where $\Re(t) = \sum_{j \in \mathbb{Z}} \frac{e^{itj^2} - 1}{ij^2}$ is a complex version of Riemann's non-differentiable function.

- Graph on $[0, 2\pi]$ of Riemann's function $\mathfrak{R}(t) = \sum_{j \in \mathbb{Z}} \frac{e^{itj^2} - 1}{ij^2}$:



- \mathfrak{R} satisfies the multifractal formalism of Frisch-Parisi (Jaffard 96) is intermittent (Boritchev-Eceizabarrena-Da Rocha 19), its graph has no tangents (at the end Riemann was right!!) and has Hausdorff dimension $\leq \frac{4}{3}$ (Eceizabarrena 19)
- The theorem gives a non-obvious non-linear geometric interpretation for Riemann's function.

**THANK YOU FOR YOUR
ATTENTION**