A NEW APPROACH TO HARDY'S UNCERTAINTY PRINCIPLE

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Hardy's Uncertainty Principle (Cowling, Price)

Define for $\xi \in \mathbb{R}^n$

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\, dx$$

Assume that

$$f(x)e^{\frac{|x|^2}{\alpha}} \quad \in L^2$$

$$\widehat{f}(\xi)e^{\frac{|\xi|^2}{\beta}} \quad \in L^2$$

- $\alpha\beta \leq 4 \quad \Longrightarrow \quad f \equiv 0$

Using L^∞ instead of L^2

- $\alpha\beta = 4 \quad \Longrightarrow \quad f = ae^{-x^2/2}$

Free Schrödinger Equation

S.E. $\begin{cases} \partial_t u = i\Delta u & x \in \mathbb{R}^n \quad t \in \mathbb{R} \\ u(0) = u_0 \end{cases}$

$$u(x,t) = \frac{1}{(it)^{n/2}} \int_{\mathbb{R}^n} e^{i\frac{|x-y|^2}{4t}} u_0(y)dy$$

$$= \frac{1}{(it)^{n/2}} e^{i\frac{|x|^2}{4t}} \int_{\mathbb{R}^n} e^{-i\frac{x}{2t}\cdot y} e^{i\frac{|y|^2}{4t}} u_0(y)dy$$

$$f(y) = e^{i\frac{|y|^2}{4t}} u_0(y),$$

$$\widehat{f}(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-ixy} f(y)dy.$$

Hence

$$u(0)e^{\frac{|y|^2}{\alpha}} \in L^2 \iff f e^{\frac{|y|^2}{\alpha}} \in L^2$$

$$u(T)e^{\frac{|x|^2}{\beta}} \in L^2 \iff \widehat{f}\left(\frac{x}{2T}\right) e^{\frac{|x|^2}{\beta}} \in L^2.$$

Hardy's uncertainty principle:

$$\alpha\beta < 4T \implies u \equiv 0$$

Schrödinger Equation with a Potential

$$u = u(x,t)$$

$$\partial_t u = i(\Delta + V)u \qquad x \in \mathbb{R}^n$$

$$\left.\begin{array}{l} \bullet \quad e^{\frac{|x|^2}{\alpha}} u(0) \in L^2 \\ \bullet \quad e^{\frac{|x|^2}{\beta}} u(T) \in L^2 \\ \bullet \quad \alpha\beta \text{ small enough} \end{array}\right\} \Longrightarrow \quad u \equiv 0$$

Theorem 1.– $u \in \mathcal{C}\left([0,T] : L^2(\mathbb{R}^n)\right)$ solution of

$$\partial_t u = i(\Delta + V)u \qquad x \in \mathbb{R}^n \quad t \in [0,T].$$

$$u(0)e^{\frac{|x|^2}{\alpha}} \in L^2 \qquad ; \qquad u(T)e^{\frac{|x|^2}{\beta}} \in L^2,$$

and $\alpha\beta < 4T$, then $u \equiv 0$.

Hypothesis on the potential:

H1 $\quad V = V_0(x) +$ perturbation with a gaussian decay
V_0 real and bounded

H1* $\quad V = V(x,t)$

$$\lim_{R\to\infty} \int_0^T \sup_{|x|\geq R} |V| dt = 0$$

Theorem 2.–There exists a non-trivial $u \in \mathcal{C}\left(\mathbb{R} : L^2(\mathbb{R}^n)\right)$ solution of

$$\partial_t u = i(\Delta + V)u \qquad x \in \mathbb{R}^n \quad t \in \mathbb{R},$$

with $V = V(x,t) \in \mathbb{C}$ and $|V(x,t)| \leq \frac{C}{1+|x|^2}$, such that

$$u(0)e^{\frac{|x|^2}{\alpha}} \in L^2 \qquad ; \qquad u(1)e^{\frac{|x|^2}{\beta}} \in L^2,$$

and $\alpha\beta = 4$.

Non-Linear Schrödinger Equation

Theorem 3.– $u_1, u_2 \in \mathcal{C}\left([0,T] : H^k(\mathbb{R}^n)\right)$ strong solutions of

$$\partial_t u = i\left(\Delta u + F(u,\overline{u})\right) \qquad x \in \mathbb{R}^n \quad t \in [0,T].$$

$$k > \frac{n}{2} \quad ; \quad F \text{ regular} \quad ; \quad F(0) = \partial_u F(0) = \partial_{\overline{u}} F(0) = 0.$$

Assume

$$e^{\frac{|x|^2}{\beta}}\left(u_1(0) - u_2(0)\right) \in L^2,$$

$$e^{\frac{|x|^2}{\alpha}}\left(u_1(T) - u_2(T)\right) \in L^2,$$

$$\alpha\beta < 4T.$$

Then $u_1 \equiv u_2$.

About the proofs in the constant coefficient case (key words):

- Uncertainty principle:
  - Positive Commutators.
- Hardy’s theorem:
  - Analyticity, Cauchy-Riemann equations, Log convexity.
  - Liouville’s theorem.

Uncertainty Principle

$$\mathcal{S} \text{ symmetric} \quad \langle \mathcal{S}f, f\rangle \quad = \quad \langle f, \mathcal{S}f\rangle$$

$$\mathcal{A} \text{ skewsymmetric} \quad \langle \mathcal{A}f, f\rangle \quad = \quad -\langle f, \mathcal{A}f\rangle$$

$$\big\langle(\mathcal{S}+\mathcal{A})f, (\mathcal{S}+\mathcal{A})f\big\rangle = \langle \mathcal{S}f, \mathcal{S}f\rangle + \langle \mathcal{A}f, \mathcal{A}f\rangle + \big\langle(\mathcal{S}\mathcal{A}-\mathcal{A}\mathcal{S})f, f\big\rangle$$

Hence

$$\big\langle(\mathcal{A}\mathcal{S}-\mathcal{S}\mathcal{A})f, f\big\rangle \leq \|\mathcal{S}f\|_{L^2}^2 + \|\mathcal{A}f\|_{L^2}^2$$

$$\mathbf{Sf} = \mathbf{xf} \qquad ; \qquad \mathcal{A}\mathbf{f} = \mathbf{f}'$$

$$\mathcal{A}\mathbf{S} - \mathbf{S}\mathcal{A} = \frac{\mathbf{d}}{\mathbf{dx}}\mathbf{x} - \mathbf{x}\frac{\mathbf{d}}{\mathbf{dx}} = \mathbb{1}.$$

$$\|f\|_{L^2}^2 \leq \|xf\|_{L^2}^2 + \|f'\|_{L^2}^2 \qquad f(x) = e^{-\frac{x^2}{2}}.$$

$$\mathcal{A} \hookrightarrow \lambda\mathcal{A} \qquad ; \qquad S \hookrightarrow \frac{1}{\lambda} S$$

$$\Big\langle (\mathcal{A}S - S\mathcal{A})f, f \Big\rangle \leq 2\|Sf\|_{L^2}\|\mathcal{A}f\|_{L^2}$$

$$\mathcal{A} \hookrightarrow \mathcal{A} - \Big\langle \mathcal{A}f, \mathcal{A}f \Big\rangle \mathbb{1} \quad ; \quad S \hookrightarrow S - \langle Sf, Sf \rangle \mathbb{1}$$

Log-Convexity (an abstract lemma)

S an $\mathcal{A}$ as before

$$\partial_t v = (S + \mathcal{A})v$$

$$H(t) = \langle v, v \rangle$$

$$\begin{aligned}\dot{H}(t) &= \langle v_t, v \rangle + \langle v, v_t \rangle \\ &= \big\langle (S + \mathcal{A})v, v \big\rangle + \big\langle v, (\mathcal{A} + S)v \big\rangle \\ &= 2\langle Sv, v \rangle.\end{aligned}$$

$$\begin{aligned}\ddot{H}(t) &= 2\big\langle Sv_t, v \big\rangle + 2\big\langle Sv, v_t \big\rangle \\ &= 2\big\langle (S + \mathcal{A})v, Sv \big\rangle + 2\big\langle Sv, (S + \mathcal{A})v \big\rangle \\ &= 4\langle Sv, Sv \rangle + 2\big\langle (S\mathcal{A} - \mathcal{A}S)v, v \big\rangle\end{aligned}$$

$$
\begin{aligned}
(\lg H(t))^{\cdot\cdot} &= \left(\frac{\dot{H}}{H}\right)^{\cdot} = \frac{\ddot{H}H - \dot{H}^2}{H^2} \\
&= \frac{1}{\langle v, v\rangle}\left\{4\langle Sv, Sv\rangle\langle v, v\rangle - 4\langle Sv, v\rangle^2 + 2\big\langle(S\mathcal{A} - \mathcal{A}S)v, v\big\rangle\right\} \\
&\geq 2\big\langle(S\mathcal{A} - \mathcal{A}S)v, v\big\rangle
\end{aligned}
$$

Hence if $2\big\langle(S\mathcal{A} - \mathcal{A}S)v, v\big\rangle \geq 0$ then

$$
H(t) \leq H(0)^{1-t}H(1)^t
$$

More generally if

$$
2\big\langle(S\mathcal{A} - \mathcal{A}S)v, v\big\rangle \geq \psi(t)\langle v, v\rangle,
$$

then

$$
H(t)e^{-B(t)} \leq H(0)^{1-t}H(1)^t,
$$

with

$$
\ddot{B} = \Psi(t), \quad B(0) = B(1) = 0.
$$

A Particular Example

- $\partial_t u = i\Delta u$

- $e^{\frac{|x|^2}{2}} u = v$

- $H(t) = \|v(t)\|_{L^2}^2 = \langle v(t), v(t) \rangle$

- $$\begin{aligned} \partial_t v &= \left(e^{\frac{|x|^2}{2}} i\Delta e^{-\frac{|x|^2}{2}} \right) v \\ &= i \sum_j e^{\frac{|x|^2}{2}} \partial_j^2 e^{-\frac{|x|^2}{2}} v \\ &= i \sum_j e^{\frac{|x|^2}{2}} \partial_j e^{-\frac{|x|^2}{2}} e^{\frac{|x|^2}{2}} \partial_j e^{-\frac{|x|^2}{2}} v \\ &= i \sum_j (x_j - \partial_j)(x_j - \partial_j) v \\ &= i \left(|x|^2 - 2x \cdot \nabla - d + \Delta \right) v \qquad \text{(C–R)} \end{aligned}$$

$$\partial_t v = (S + \mathcal{A})v \quad ; \quad Sv = -i(2x \cdot \nabla + d)v \quad ; \quad \mathcal{A}v = i(\Delta + |x|^2)$$

$$S\mathcal{A} - \mathcal{A}S = -4\Delta + 4|x|^2$$

$$\big\langle S\mathcal{A} - \mathcal{A}Sv, v\big\rangle = 4(\|\nabla v\|^2 + \|xv\|^2) \geq 4\langle v, v\rangle.$$

Hence

$$(\lg H(t))^{\cdot\cdot} \geq 8$$

and

$$H(t) \leq H(0)^{1-t} H(1)^t$$

- Therefore u has a gaussian decay for $0 < t < 1$!!!

Remark.– If u_0 has gaussian decay then $e^{it\Delta}u_0$ does not necessarily have it for $t > 0$ $\left(u_0 = (\operatorname{sig} x)e^{-|x|^2}\right)$.

Another Particular Example (A Lower Bound Carleman, Holmgren, Isakov)

We define

$$H(t) = \left\| e^{\mu|x+e_1 b(t)|^2} u(t) \right\|_{L^2}^2 e^{-B(t)} \qquad e_1 = (1, 0, \ldots, 0)$$

with $b(0) = b(1) = 0 = B(0) = B(1)$ and

$$\ddot{B} = \frac{\left(\ddot{b}\right)^2}{32\mu}.$$

Then $\lg H$ is convex.

The optimal choice is

$$\begin{aligned} b(t) &= Rt(1-t) \qquad b(1/2) = \frac{R}{4} \\ B(t) &= \frac{R^2}{8\mu} t(1-t) \qquad B(1/2) = \frac{R^2}{32\mu} \end{aligned}$$

$$H(1/2) \leq H(0)^{1/2} H(1)^{1/2}$$

$$\int_{\mathbb{R}^n} |u(x)|^2 e^{\mu |x|^2} e^{\varphi(R)} \, dx < +\infty,$$

with

$$\varphi(R) = \frac{\mu}{2} R \, e_1 \cdot x + \frac{\mu}{8} R^2 \big(\mu - \frac{1}{4\mu}\big).$$

Two possibilities:

a) $\mu - \frac{1}{4\mu} \geq 0 \quad \text{then} \quad u \equiv 0 \quad (\mu \geq 1/2);$

b) $\mu - \frac{1}{4\mu} < 0$. Integrating in R we conclude that there exists $a_1(t) > \mu = a_0(t)$ if $t \in (0,1)$ and

$$a_1(0) = a_1(1) = \mu$$

such that

$$\int_{\mathbb{R}^n} |u(x)|^2 e^{a_1(t)|x|^2}\, dx < H(0)^{1-t} H(1)^t.$$

Self-improvement!!

Iteration

Then we repeat the process and either we conclude that $u \equiv 0$ or that there exists a sequence of functions $a_k(t)$ with

$$a_1(0) = a_1(1) = \mu$$

and

$$a_k(t) > a_{k-1} > \cdots > a_0(t) = \mu \quad k \in \mathbb{N} \quad t \in (0,1),$$

with

$$\lim a_k(t) = a(t)$$

a solution of the ode

$$\ddot{a} - \frac{3}{2}\frac{\dot{a}^2}{a} + 32a^3 = 0.$$

As a consequence

Theorem 4.–

$$H(t) = \int_{\mathbb{R}^n} |u(x,t)|^2 e^{a(t)|x|^2}\,dx < C\big(H(0) + H(1)\big).$$

with

$$a(t) = \frac{R}{4\big(1 + R^2(t-1/2)^2\big)} \quad \text{for some}\, R > 0,$$

and

$$a(0) = \frac{R}{(4+R^2)} = \mu \leq \frac{1}{4}!!.$$

Above R is the smallest R such that $\frac{R}{(4+R^2)} = \mu$.

Misleading algebraic manipulations

Define

$$H(t) = \left\langle e^{a(t)\frac{|x|^2}{2}} u, u \right\rangle \qquad t \in [-1, 1]$$

Then H is (formally) $1/a$-log convex if a solves

$$(*) \qquad \ddot{a} - \frac{3}{2}\frac{\dot{a}^2}{a} + 32a^3 = 0.$$

If $a(x)$ solves $(*)$ then $a_R(x) = Ra(Rx)$ is also a solution. This easily leads to a contradiction!!!

Lemma.– Assume that

$$u \in L^{\infty}\left([0,1]\,,\,L^2(\mathbb{R}^n)\right) \cap L^2\left([0,1]\,,\,H^1(\mathbb{R}^n)\right)$$

satisfies

$$\partial_t u = (A+iB)(\Delta+V)u, \qquad x\in\mathbb{R}^n, \quad t\in[0,1],$$

$A>0\,,\ B\in\mathbb{R}\,,$ and $\|V\|_{L^\infty}\le M_1$. Call $H(t)=\left\|e^{\frac{|x|^2}{2}}u(t)\right\|^2$.

Then

- $H(t)\le e^{c\left(A^2+B^2\right)M_1^2}H(0)^{1-t}H(1)^t.$

Moreover

- $\left\|\sqrt{t(1-t)}\,(|\nabla u|+|x||u|)\,e^{\frac{|x|^2}{2}}\right\|_{L^2}\le C\,(H(0)+H(1))\,.$

Remarks.–

- Convexity remains in the parabolic setting.

- Gaussian decay remains with positive diffusion. The decay rates are worse for positive times. Hence $e^{|x|^2}$ has to be changed into $e^{|x|^a}$ $a < 2$.

Lemma 2.– $\mathcal{S}$ is a symmetric operator, $\mathcal{A}$ is skew–symmetric, both are allowed to depend on the time variable, G is a positive function, $f(x,t)$ is a reasonable function,

$$H(t) = (f,f)\ \ ,\ \ D(t) = (\mathcal{S}f,f)\ \ ,\ \ \partial_t\mathcal{S} = \mathcal{S}_t \text{ and } N(t) = \frac{D(t)}{H(t)}$$

Then,

$$\partial_t^2 H = 2\partial_t \mathrm{Re}\ (\partial_t f - \mathcal{A}f - \mathcal{S}f, f) + 2\left(\mathcal{S}_t f + [\mathcal{S},\mathcal{A}]f, f\right)$$
$$+ \left\|\partial_t f - \mathcal{A}f + \mathcal{S}f\right\|^2 - \left\|\partial_t f - \mathcal{A}f - \mathcal{S}f\right\|^2$$

and

$$\dot{N}(t) \geq \left(\mathcal{S}_t f + [\mathcal{S},\mathcal{A}]f, f\right)/H - \left\|\partial_t f - \mathcal{A}f - \mathcal{S}f\right\|^2/(2H).$$

Moreover, if

$$\left|\partial_t f - \mathcal{A}f - \mathcal{S}f\right| \le M_1|f| + G, \quad \text{in } \mathbb{R}^n \times [0,1], \quad \mathcal{S}_t + [\mathcal{S}, \mathcal{A}] \ge -M_0$$

and

$$M_2 = \sup_{[0,1]} \|G(t)\| / \|f(t)\|$$

is finite, then $\log H(t)$ is "logarithmically convex" in $[0,1]$ and there is a universal constant N such that

$$H(t) \le e^{N\left(M_0+M_1+M_2+M_1^2+M_2^2\right)} H(0)^{1-t} H(1)^t, \quad \text{when } 0 \le t \le 1.$$

KdV equation $u = u(x,t)$

$$\partial_t u = \partial_x^3 u + u^p \partial_x u \quad x \in \mathbb{R}$$

$$\left.\begin{array}{l} \bullet \quad e^{\frac{|x|^{3/2}}{\alpha}} u(0) \in L^2 \\ \bullet \quad e^{\frac{|x|^{3/2}}{\beta}} u(T) \in L^2 \\ \bullet \quad \alpha\beta \text{ small enough} \end{array}\right\} \implies u \equiv 0$$

Further Results

- Morgan's theorem (Bonami, Demange, Jaming).
- Blow up profiles fon non-linear dispersive equations (Meshkov counterexample).
- Sharp version of Hardy's theorem (joint work with M. Cowling).

Heat equation $u = u(x,t)$

$$\partial_t u = (\Delta + V)u \qquad x \in \mathbb{R}^n$$

$$\left.\begin{array}{l} \bullet \quad u(0) \in L^2 \\ \bullet \quad e^{\frac{|x|^2}{\beta}} u(T) \in L^2 \\ \bullet \quad \beta \text{ small enough} \end{array}\right\} \Longrightarrow \quad u \equiv 0$$

$$\text{H.E.} \qquad \begin{cases} \partial_t u = \Delta u \\ u(0) = u_0. \end{cases}$$

$$u(x,t) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-t|y|^2 + ixy} \widehat{u}_0(y) dy.$$

$$u(T) = f \qquad \widehat{u}(T) = e^{-T|y|^2} \widehat{u}_0.$$

Then

$$u(0) \in L^2 \iff \widehat{u}(T) e^{T|y|^2} \in L^2,$$

$$u(T) e^{\frac{|x|^2}{\beta}} \in L^2 \iff u(T) e^{\frac{|x|^2}{\beta}} \in L^2.$$

Hardy's uncertainty principle:

$$\beta < 2\sqrt{T} \quad \Longrightarrow u \equiv 0$$

Theorem 2.– $u \in L^\infty\big([0,T] : L^2(\mathbb{R}^n)\big) \cap L^2\big([0,T] : H^1(\mathbb{R}^n)\big)$ solution of

$$\partial_t u = (\Delta + V)u \qquad x \in \mathbb{R}^n \quad t \in [0,T].$$

$$u(0) \in L^2 \qquad ; \qquad u(T)e^{\frac{|x|^2}{\beta}} \in L^2,$$

and $\beta < \sqrt{T}$. Then $u \equiv 0$.

Hypothesis on the potential:

H2 $V = V(x,t)$ bounded in $\mathbb{R}^n \times [0,T]$.

THANK YOU FOR YOUR ATTENTION