Integration of hp-adaptivity with a Two Grid Solver: Applications to Electromagnetics.

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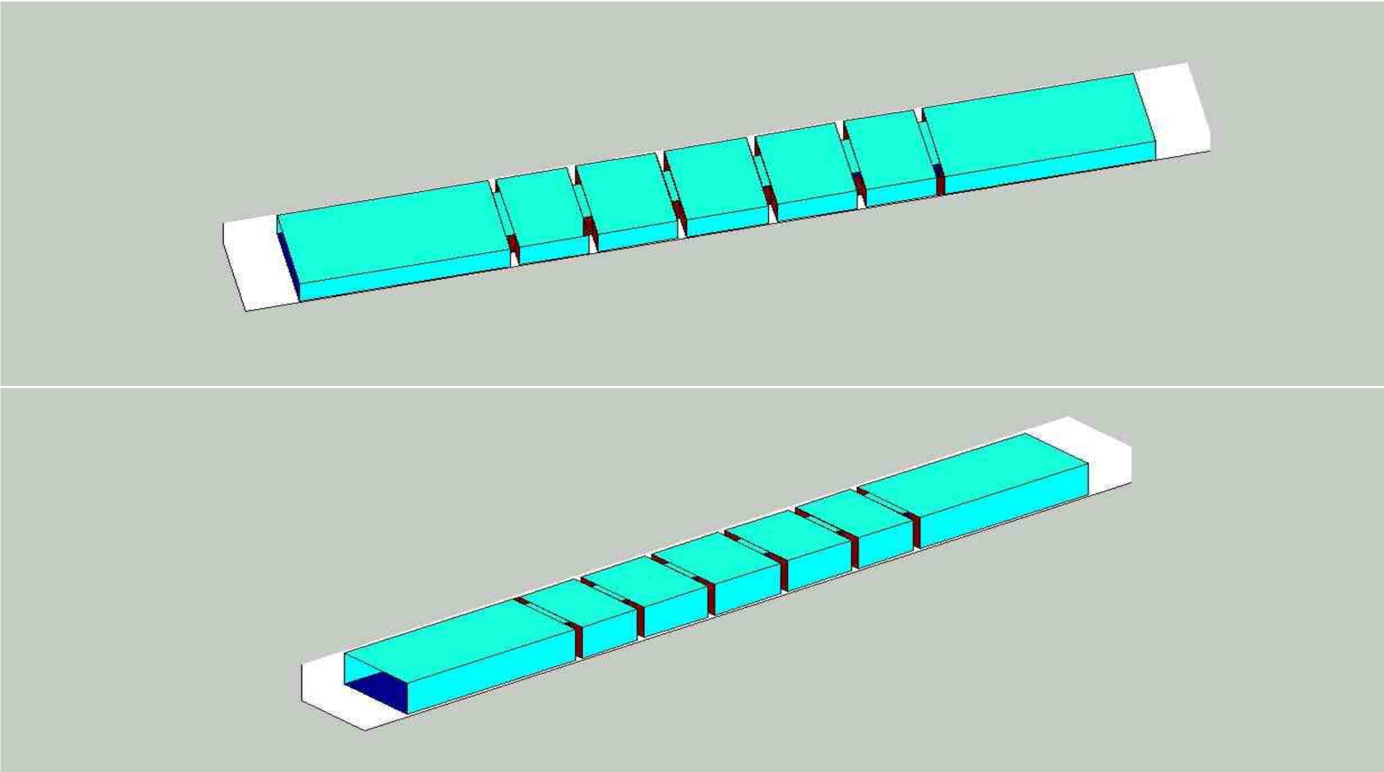
OVERVIEW

1. Overview.
2. Motivation.
3. Maxwell's Equations.
4. hp-Adaptivity.
5. The Fully Automatic *hp*-Adaptive Strategy.
6. A Two Grid Solver for SPD Problems.
7. Numerical Results.
8. A Two Grid Solver for Electromagnetics.
9. Electromagnetic Applications.
10. Conclusions and Future Work.

2. MOTIVATION

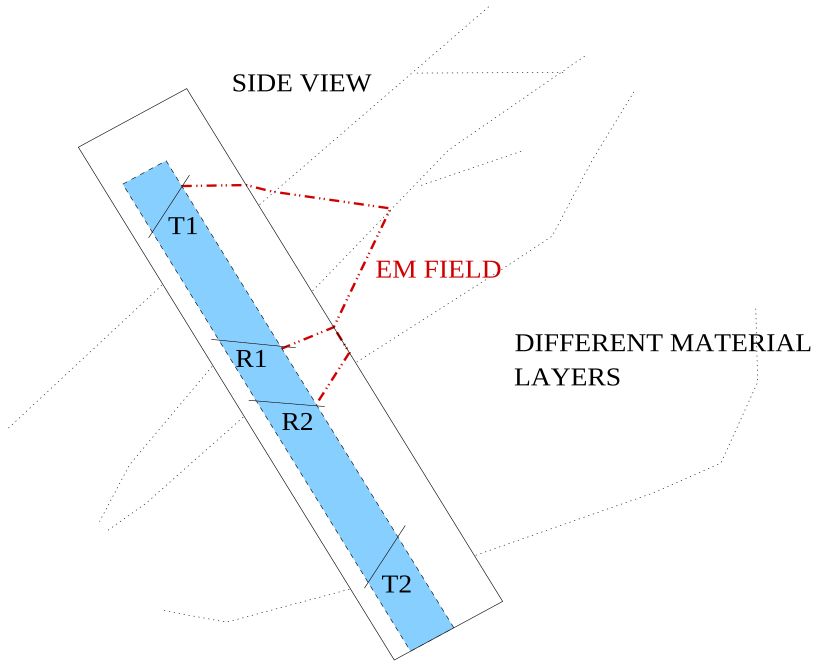
Waveguide Design



Goal: Determine electric field intensity at the ports.

2. MOTIVATION

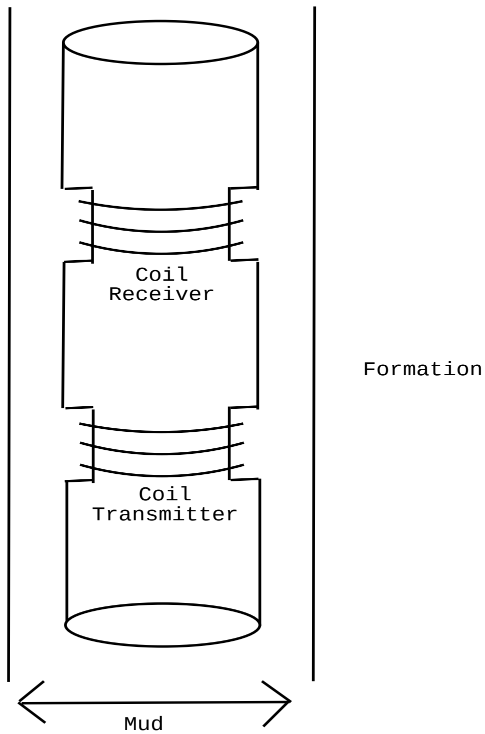
Modeling of a Logging While Drilling (LWD) electromagnetic measuring device



Goal: Determine EM field at the receiver antennas.

2. MOTIVATION

Modeling of a Logging While Drilling (LWD) electromagnetic measuring device



Simplest case:
ONE COIL TRANSMITTER

Goal: Determine EM field at the receiver antennas.

3. MAXWELL'S EQUATIONS

Time Harmonic Maxwell's Equations:

$$\nabla \times \mathbf{E} = -j\mu\omega\mathbf{H}$$
$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \sigma\mathbf{E} + \mathbf{J}^{imp}$$

Reduced Wave Equation:

$$\boldsymbol{\nabla} \times \left(\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}\right) - (\omega^2\epsilon - j\omega\sigma)\boldsymbol{E} = -j\omega\boldsymbol{J}^{imp} \, ,$$

Boundary Conditions (BC):

- **Dirichlet BC at a PEC surface:**

$$\mathbf{n} \times \boldsymbol{E}^s = -\mathbf{n} \times \boldsymbol{E}^{inc} \qquad \mathbf{n} \times \boldsymbol{E} = 0$$

- **Neumann continuity BC at a material interface:**

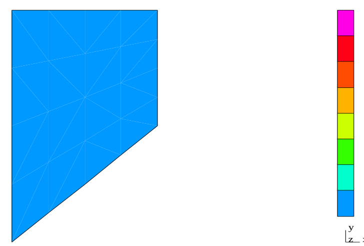
$$\mathbf{n} \times \frac{1}{\mu}\nabla \times \boldsymbol{E}^s = -\mathbf{n} \times \frac{1}{\mu}\nabla \times \boldsymbol{E}^{inc} \qquad \mathbf{n} \times \frac{1}{\mu}\nabla \times \boldsymbol{E} = -j\omega\mathbf{J}_S^{imp}$$

- **Silver Müller radiation condition at ∞:**

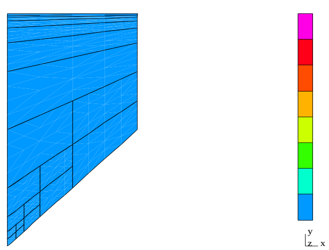
$$\mathbf{e}_r \times (\nabla \times \boldsymbol{E}^s) - jk_0 \times \boldsymbol{E}^s = O(r^{-2})$$

4. *HP*-ADAPTIVITY

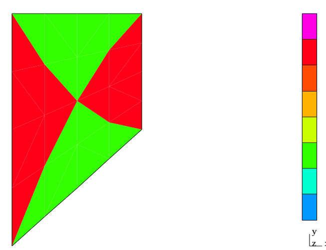
Different refinement strategies for finite elements:



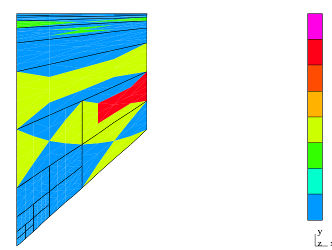
Given initial grid



***h*-refined grid**



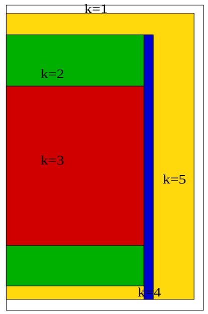
***p*-refined grid**



***hp*-refined grid**

4. *HP*-ADAPTIVITY

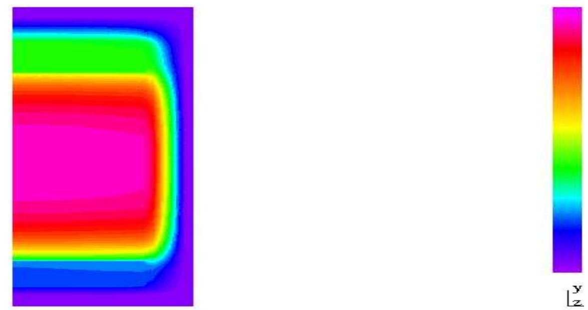
Orthotropic heat conduction example



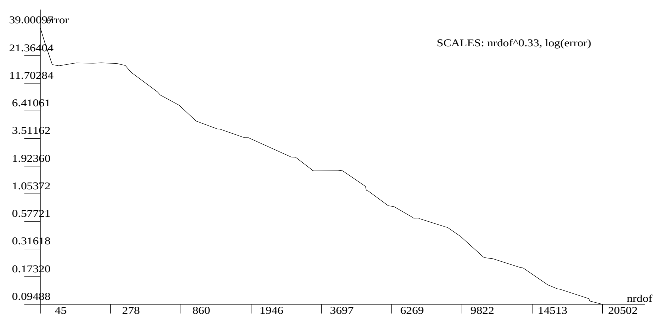
Equation: $\nabla(\mathrm{K}\nabla u) = f^{(k)}$

$$\mathrm{K} = \mathrm{K}^{(k)} = \begin{bmatrix} K_x^{(k)} & 0 \\ 0 & K_y^{(k)} \end{bmatrix}$$

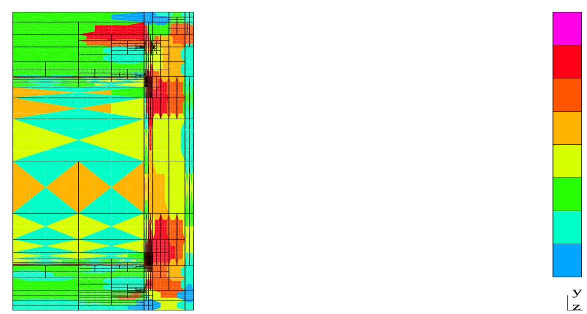
$$K_x^{(k)} = (25,\ 7,\ 5,\ 0.2,\ 0.05)$$
$$K_y^{(k)} = (25,\ 0.8,\ 0.0001,\ 0.2,\ 0.05)$$



Solution: unknown
Boundary Conditions:
$K^{(i)}\nabla u \cdot n = g^{(i)} - \alpha^{(i)} u$



Convergence history
(tolerance error = 0.1 %)

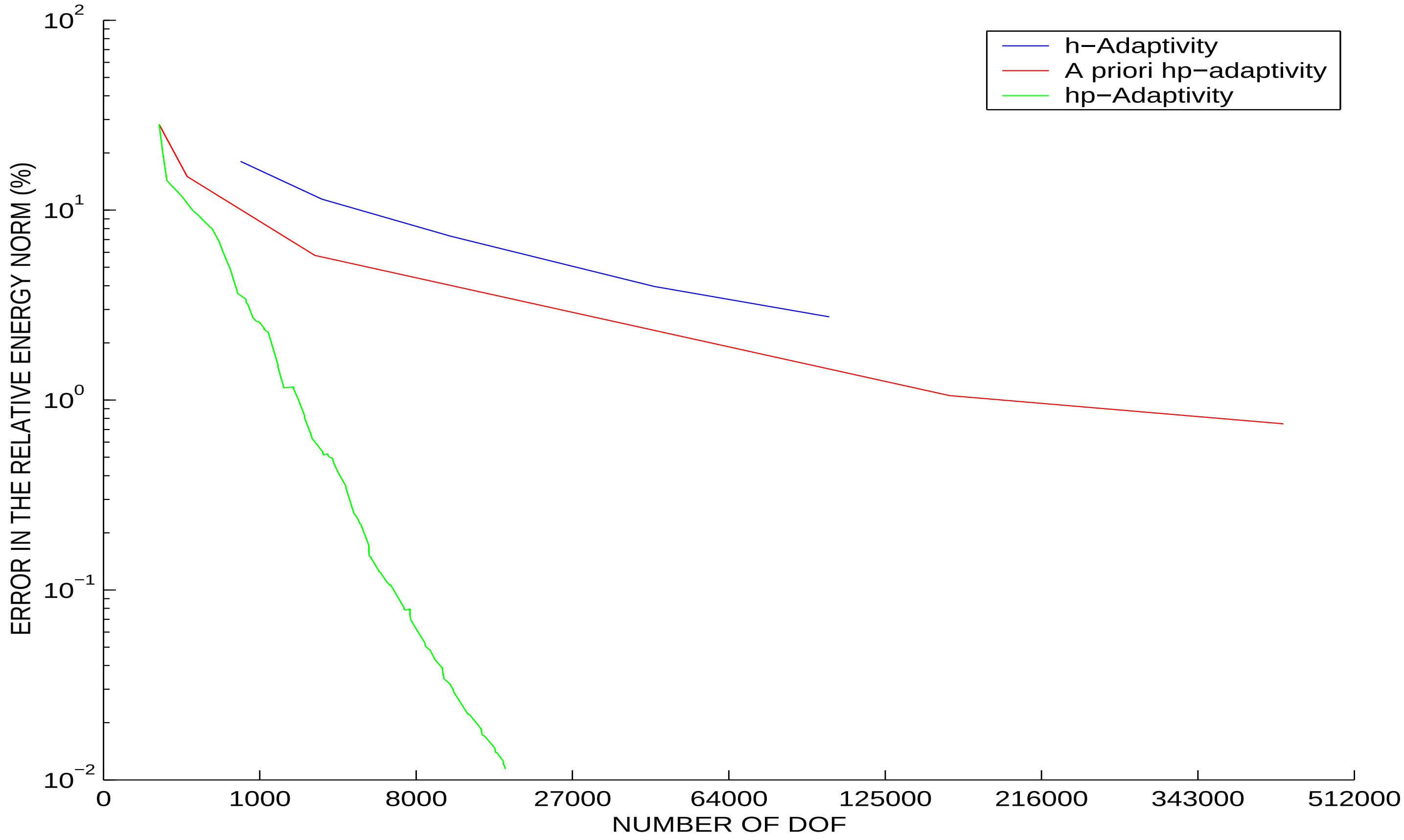


Final hp grid

4. *HP*-ADAPTIVITY

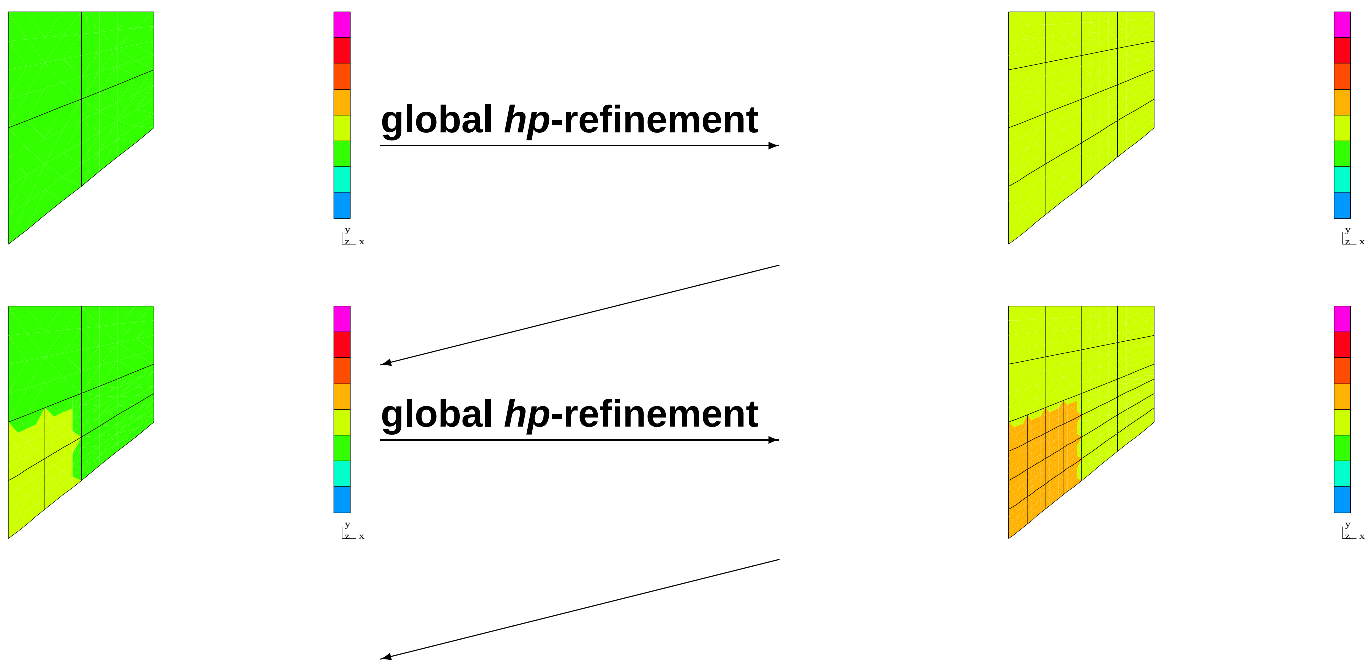
Convergence comparison

Orthotropic heat conduction example



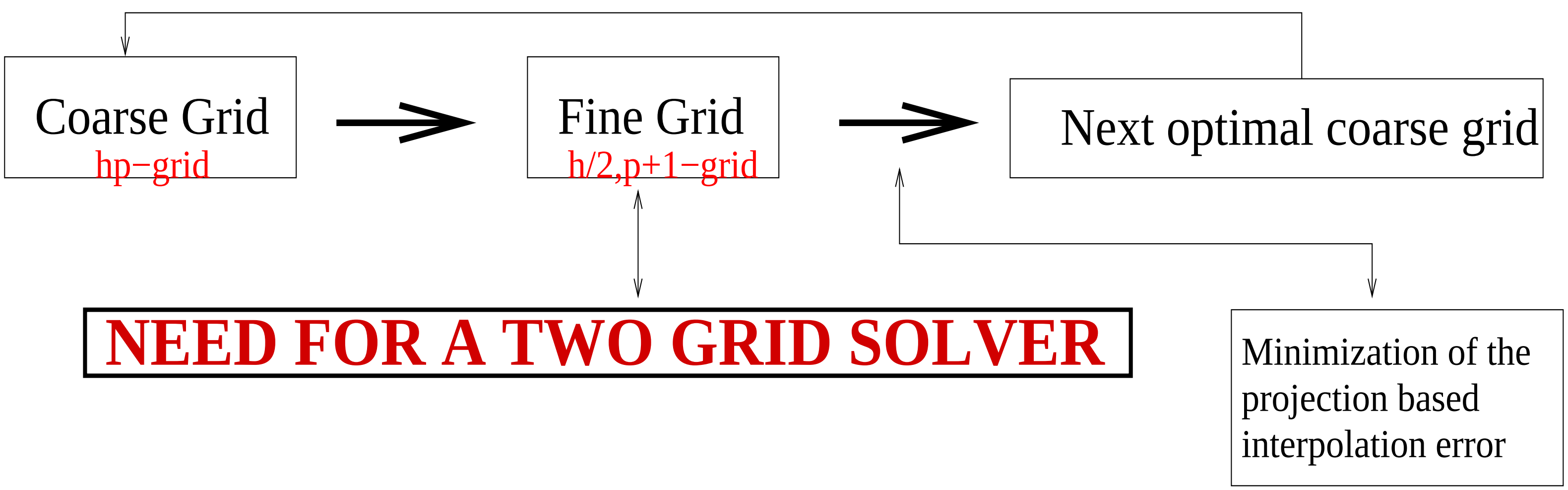
5. THE FULLY AUTOMATIC *HP*-ADAPTIVE STRATEGY

Fully automatic *hp*-adaptive strategy



5. THE FULLY AUTOMATIC *HP*-ADAPTIVE STRATEGY

Automatic *hp*-adaptivity delivers exponential convergence and enables solution of challenging EM problems



6 A TWO GRID SOLVER FOR SPD PROBLEMS

We seek x such that $Ax = b$. Consider the following iterative scheme:

$$r^{(n+1)} = [I - \alpha^{(n)} AS] r^{(n)}$$
$$x^{(n+1)} = x^{(n)} + \alpha^{(n)} S r^{(n)}$$

where S is a matrix, and $\alpha^{(n)}$ is a relaxation parameter. $\alpha^{(n)}$ *optimal* if:

$$\alpha^{(n)} = \arg\ \min \parallel x^{(n+1)} - x \parallel_A = \frac{(A^{-1} r^{(n)}, S r^{(n)})_A}{(S r^{(n)}, S r^{(n)})_A}$$

Then, we define our two grid solver as:

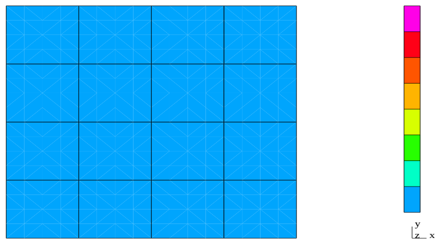
1 Iteration with $S = S_F = \sum A_i^{-1} \quad +$

1 Iteration with $S = S_C = P A_C^{-1} R$

6 A TWO GRID SOLVER FOR SPD PROBLEMS

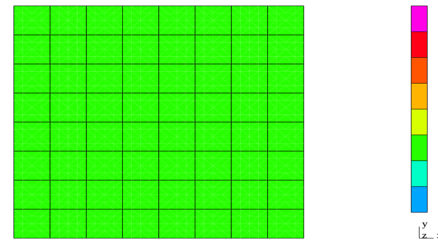
Selection of patches (for block Jacobi smoother)

Coarse Grid

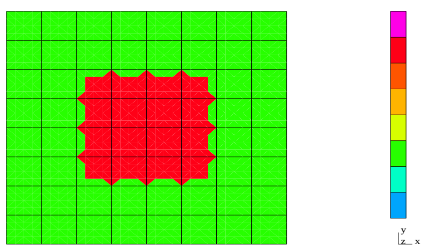


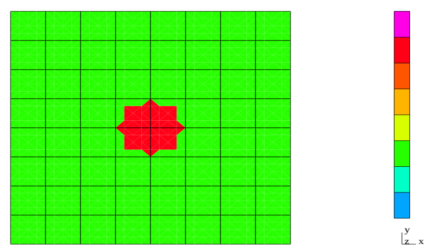
global *hp*-refinement →

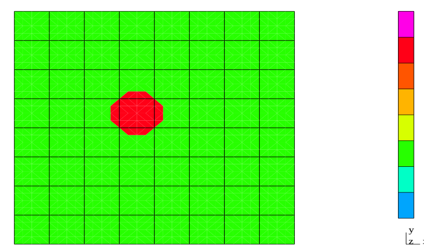
Fine Grid



Three examples of patches (blocks) for the Block Jacobi smoother:







Example 1: span of basis functions with support contained in the support of a coarse grid vertex node basis function.
Example 2: span of basis functions with support contained in the support of a fine grid vertex node basis function.
Example 3: span of basis functions corresponding to an element stiffness matrix.

6 A TWO GRID SOLVER FOR SPD PROBLEMS

Error reduction and stopping criteria

Let $e^{(n)} = x^{(n)} - x$ the error at step n, $\tilde{e}^{(n)} = [I - S_C A]e^{(n)} = [I - P_C]e^{(n)}$. Then:

$$\frac{\| e^{(n+1)} \|_A^2}{\| e^{(n)} \|_A^2} = 1 - \frac{| (\tilde{e}^{(n)}, S_F A \tilde{e}^{(n)})_A |^2}{\| \tilde{e}^{(n)} \|_A^2 \| S_F A \tilde{e}^{(n)} \|_A^2} = 1 - \frac{| (\tilde{e}^{(n)}, (P_C + S_F A) \tilde{e}^{(n)})_A |^2}{\| \tilde{e}^{(n)} \|_A^2 \| S_F A \tilde{e}^{(n)} \|_A^2}$$

Then:

$$\frac{\| e^{(n+1)} \|_A^2}{\| e^{(n)} \|_A^2} \leq \sup_e [1 - \frac{| (e, (P_C + S_F A) e)_A |^2}{\| e \|_A^2 \| S_F A e \|_A^2}] \leq C < 1$$ **(Error Reduction)**

For our stopping criteria, we want: Iterative Solver Error $\approx$ Discretization Error. That is:

$$\frac{\| e^{(n+1)} \|_A}{\| e^{(0)} \|_A} \leq 0.01$$ **(Stopping Criteria)**

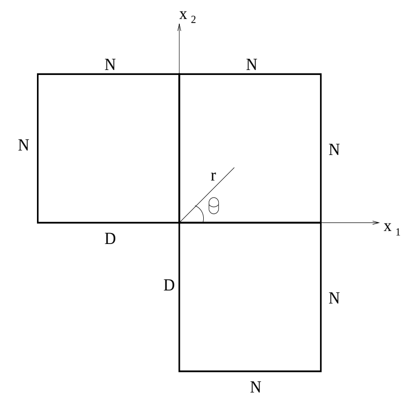
7 NUMERICAL RESULTS (TG FOR SPD)

Numerical Studies

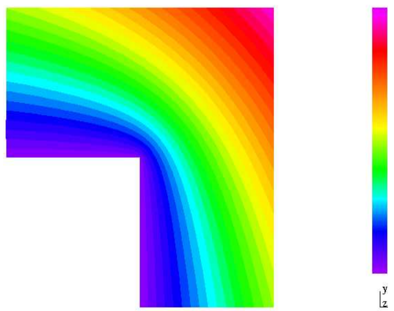
- **Examples.**
- **Importance of the relaxation parameter.**
- **Different smoothers.**
- **Error estimation.**
- **Guiding hp-adaptivity with a partially converged fine grid solution.**
- **Efficiency.**
- **Exponential convergence.**

7.1 EXAMPLES

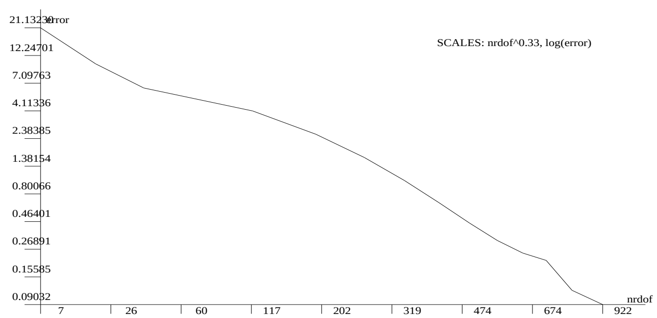
L-shape domain example



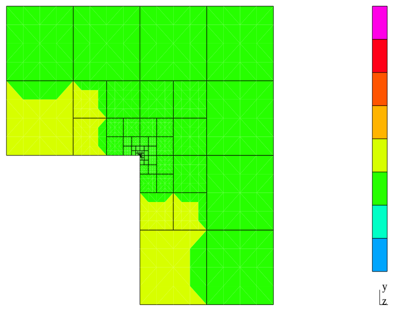
Equation: $-\Delta u = 0$
Boundary Conditions: N-Neumann, D-Dirichlet



Solution:
$u = r^{2/3} sin(2\theta/3 + \pi/3)$



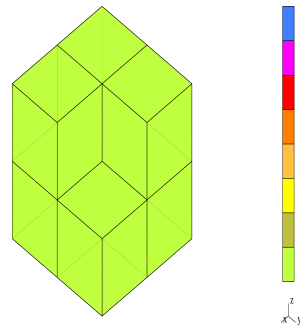
Convergence history
(tolerance error = 0.1 %)



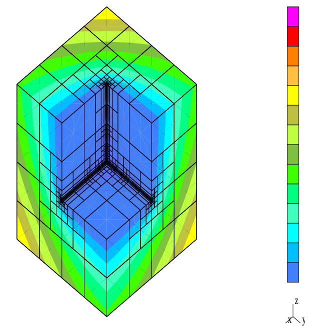
Final hp-grid

7.1 EXAMPLES

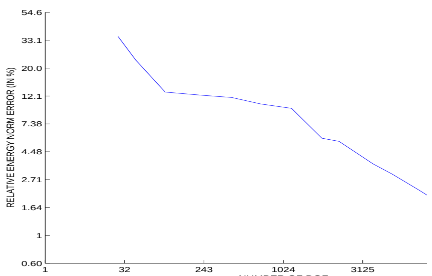
Fickera problem



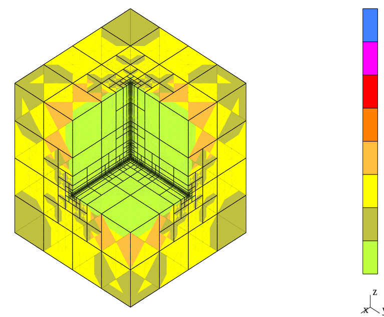
Equation: $-\Delta u = 0$
Boundary Conditions: Neumann and Dirichlet



Solution: Unknown



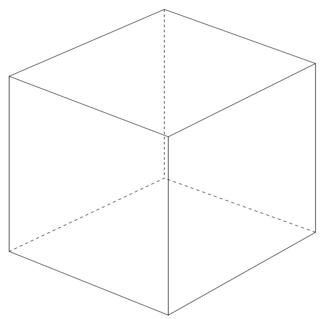
Convergence history
(tolerance error = 1 %)



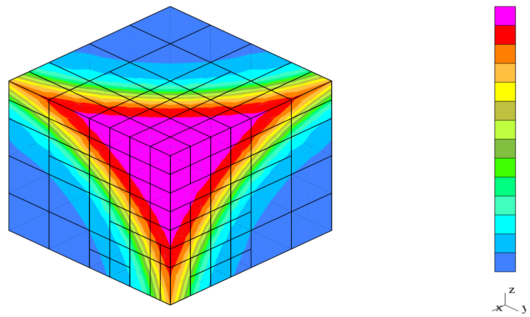
Final hp-grid

7.1 EXAMPLES

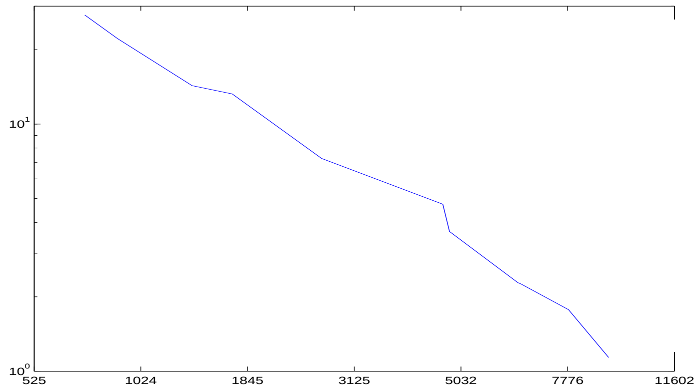
3D shock like solution example



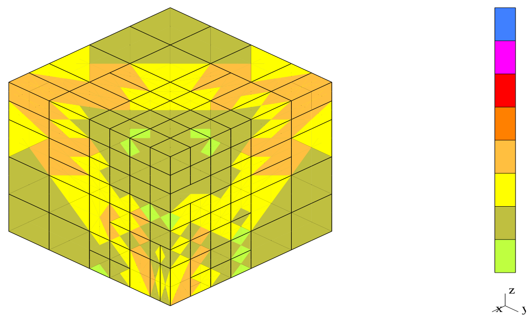
Equation: $-\Delta u = f$
Geometry: unit cube



Solution: $u = atan(20 * \sqrt{r} - \sqrt{3}))$
$r = (x - .25) * *2 + (y - .25) * *2 + (z - .25) * *2$
Dirichlet Boundary Conditions



Convergence history
(tolerance error = 1%)

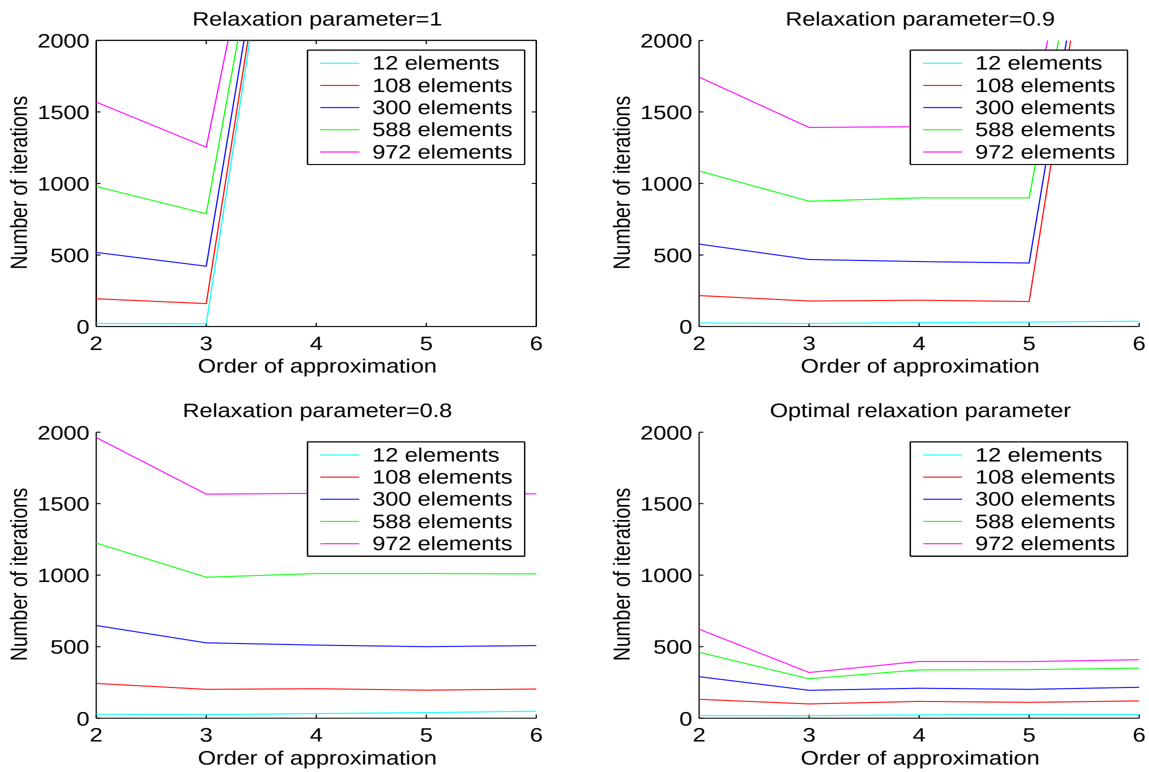


Final hp grid

7.2 IMPORTANCE OF RELAXATION PARAMETER

Relaxation parameter

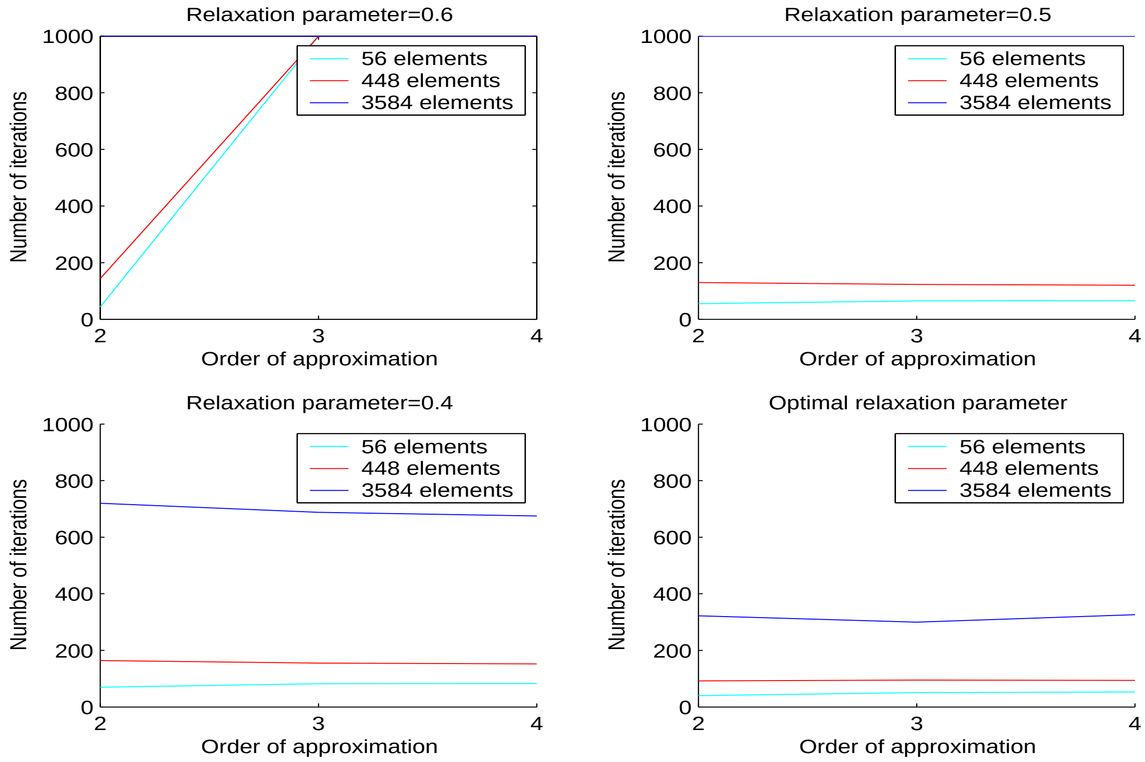
L-shape domain example (only smoothing operations)



Convergence or not, depends almost exclusively upon p.
***Convergence rate* of the method (provided that the method converges) depends almost exclusively upon h.**
The optimal relaxation guarantees faster convergence than any fixed relaxation parameter.

7.2 IMPORTANCE OF RELAXATION PARAMETER

Fichera problem

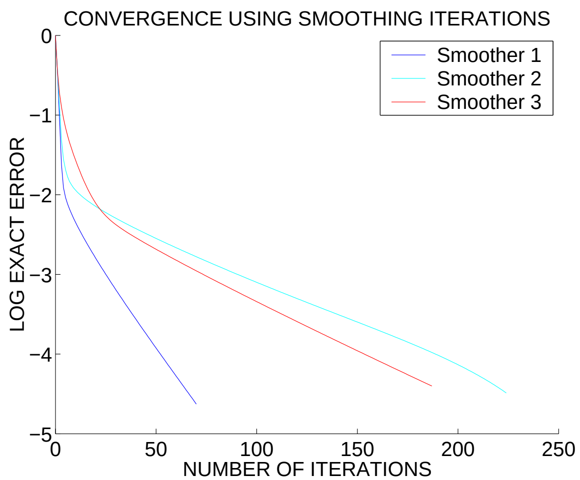


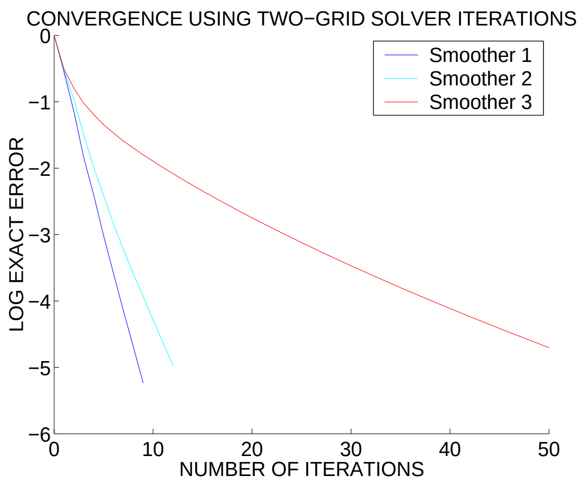
Convergence or not, depends almost exclusively upon p.
Convergence rate **of the method (provided that the method converges) depends almost exclusively upon h.**
The optimal relaxation guarantees faster convergence than any fixed relaxation parameter.

7.3 DIFFERENT SMOOTHERS

Performance of different smoothers

L-shape domain example (11837 dof)





Smoother 1: requires 16 times more memory than stiffness matrix.

Smoother 2: requires 4 times more memory than stiffness matrix.

Smoother 3: requires as much memory as the stiffness matrix.

7.4 ERROR ESTIMATION

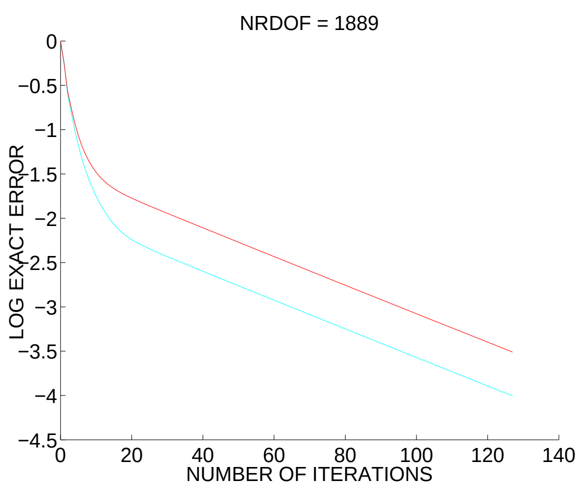
Error Estimation

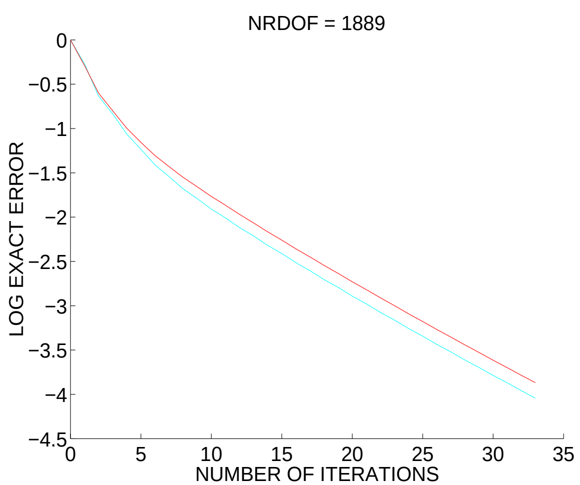
$$\frac{\| e^{(n)} \|_A}{\| e^{(0)} \|_A} = \frac{\| A^{-1} r^{(n)} \|_A}{\| A^{-1} r^{(0)} \|_A} \approx \frac{\| \alpha^{(n)} S r^{(n)} e^{(n)} \|_A}{\| \alpha^{(0)} S r^{(0)} e^{(0)} \|_A}$$

(Error Estimate)

L-shape domain (1889 dof)

Smoothing iterations only **Two grid solver iterations**





Comparing the exact error vs an estimate to the error.

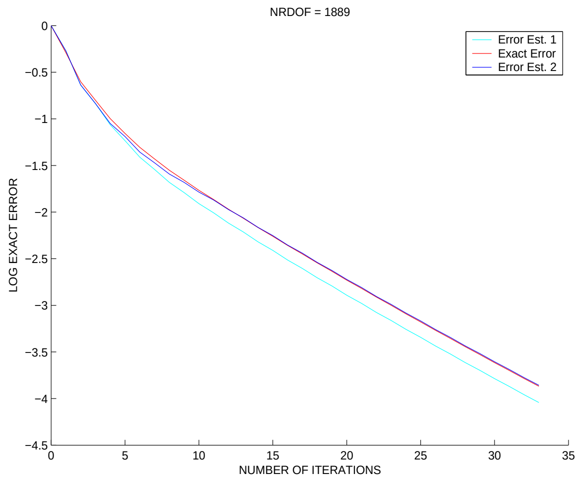
7.4 ERROR ESTIMATION

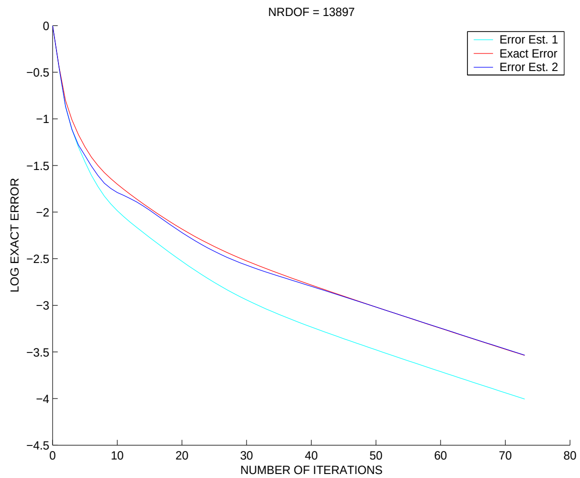
$$\frac{\| e^{(n)} \|_A}{\| e^{(0)} \|_A} \approx \frac{\| \alpha^{(n)} S_F r^{(n)} \|_A}{\| \alpha^{(0)} S_F r^{(0)} \|_A} \approx \frac{\| \alpha^{(n)} S_F r^{(n)} \|_A}{\| \alpha^{(0)} S_F r^{(0)} \|_A} * C(n)$$

(Error Estimate II)

Error estimation for 2D and 3D problems

L-shape domain (1889 dof) **Fichera problem (13897 dof)**



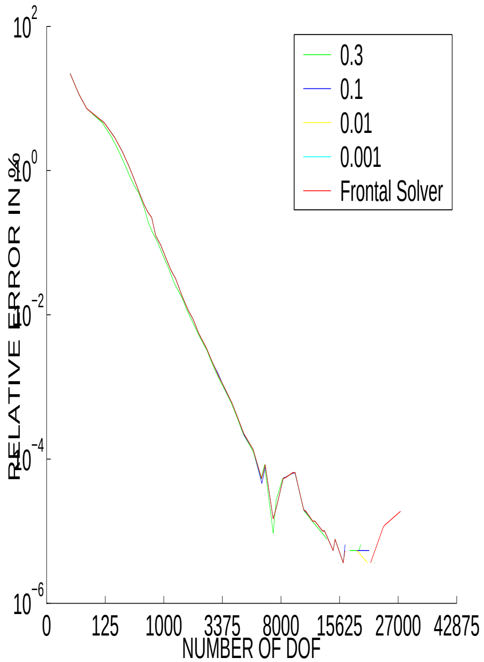


Comparing the exact error vs error estimate I vs error estimate II.

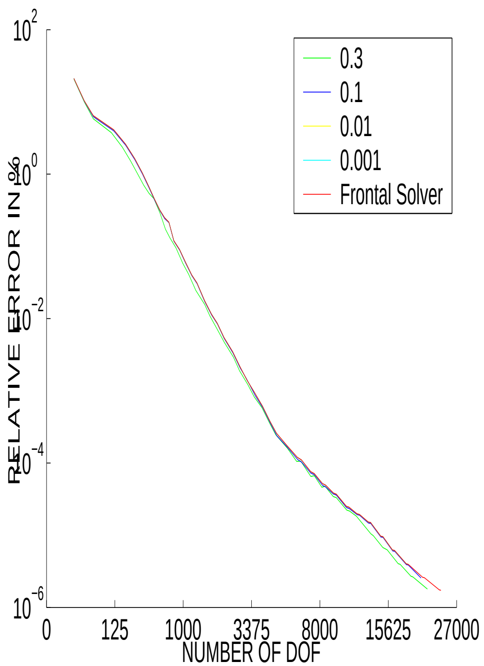
7.5 GUIDING HP-REFINEMENTS

Guiding automatic hp-refinements

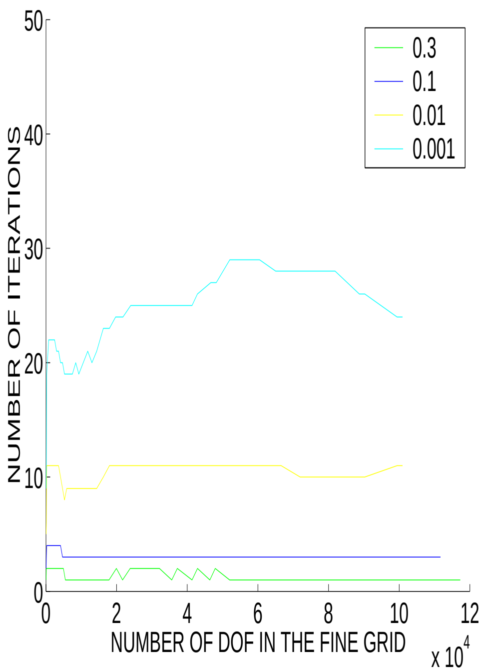
L-shape domain problem. Guiding hp-refinements with a partially converged solution.



Energy error estimate



Discretization error estimate

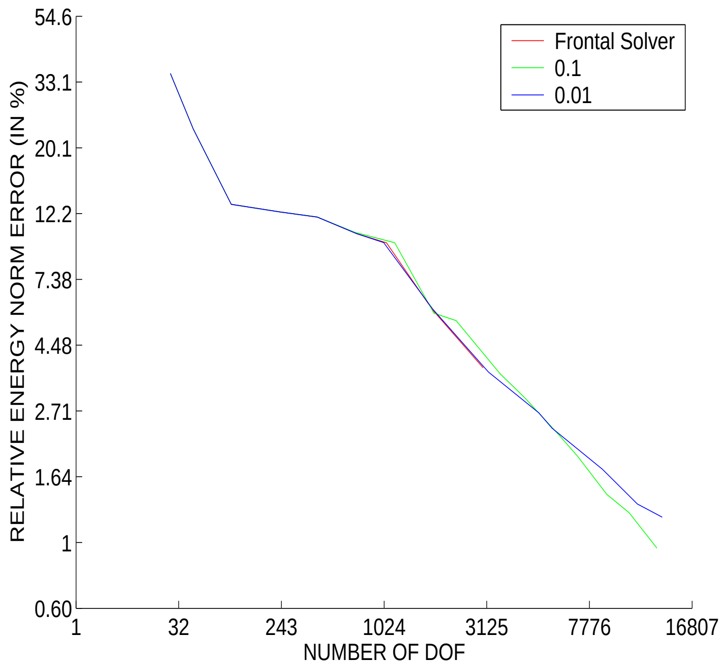


Number of iterations

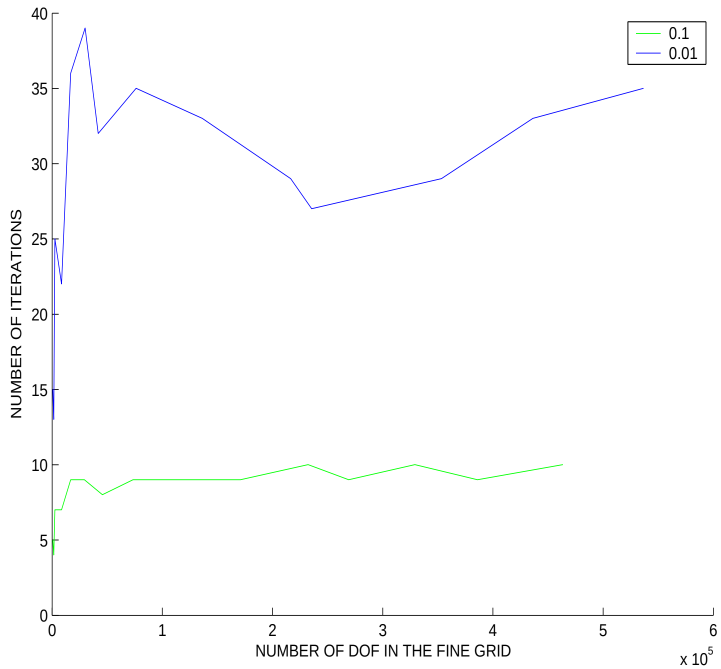
7.5 GUIDING HP-REFINEMENTS

Guiding automatic hp-refinements

Fickera problem. Guiding hp-refinements with a partially converged solution.



Discretization error estimate



Number of iterations

7.6 EFFICIENCY

Efficiency of the two grid solver

We studied scalability of the solver with respect h and p.

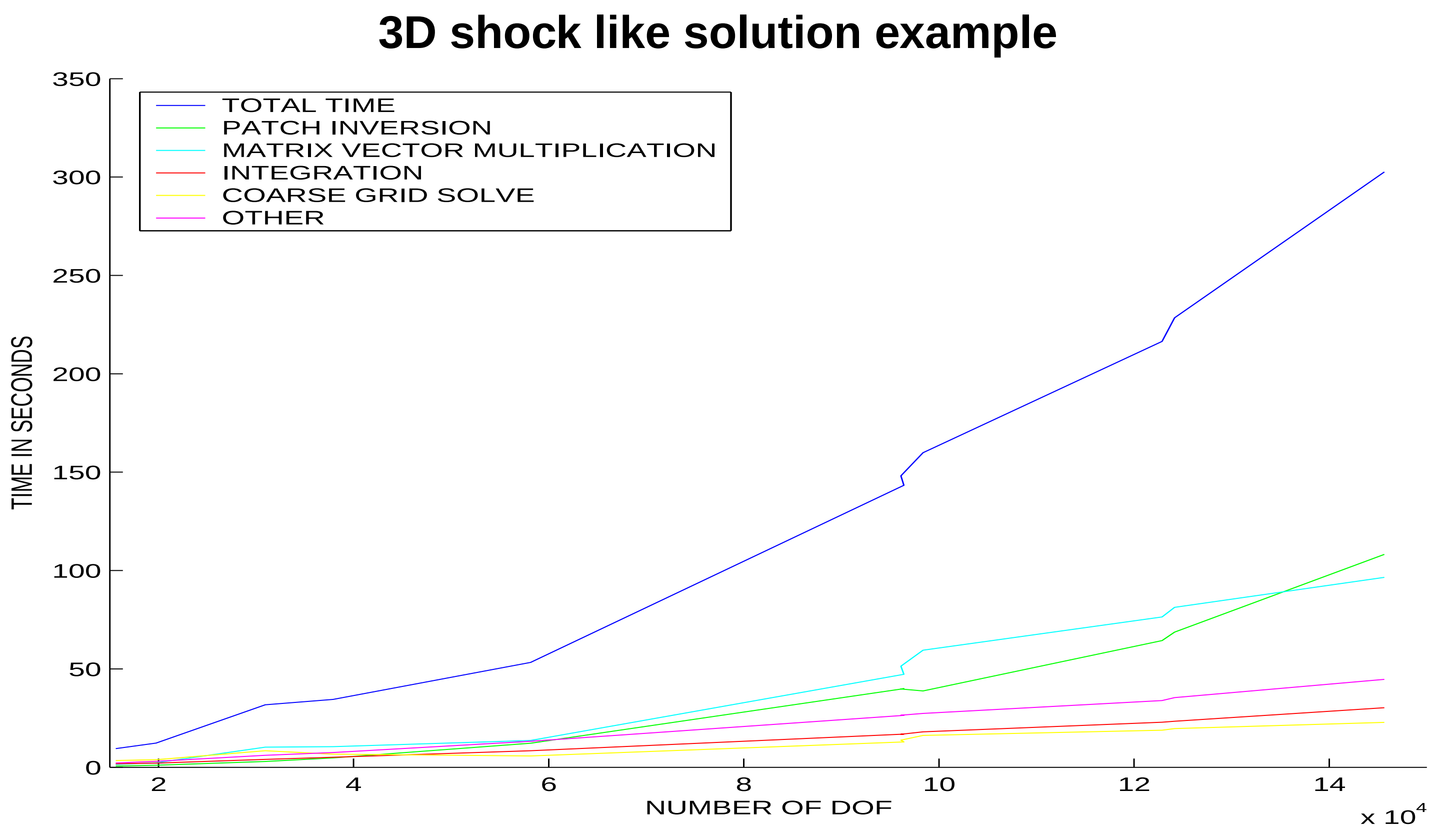
Speed = Coarse grid solve $+\mathcal{O}(p^9 N)$

We implemented an efficient solver.

- **Fast integration rules.**
- **Fast matrix vector multiplication.**
- **Fast assembling.**
- **Fast patch inversion.**
- **Fast construction of prolongation/restriction operator.**

7.6 EFFICIENCY

Performance of the two grid solver

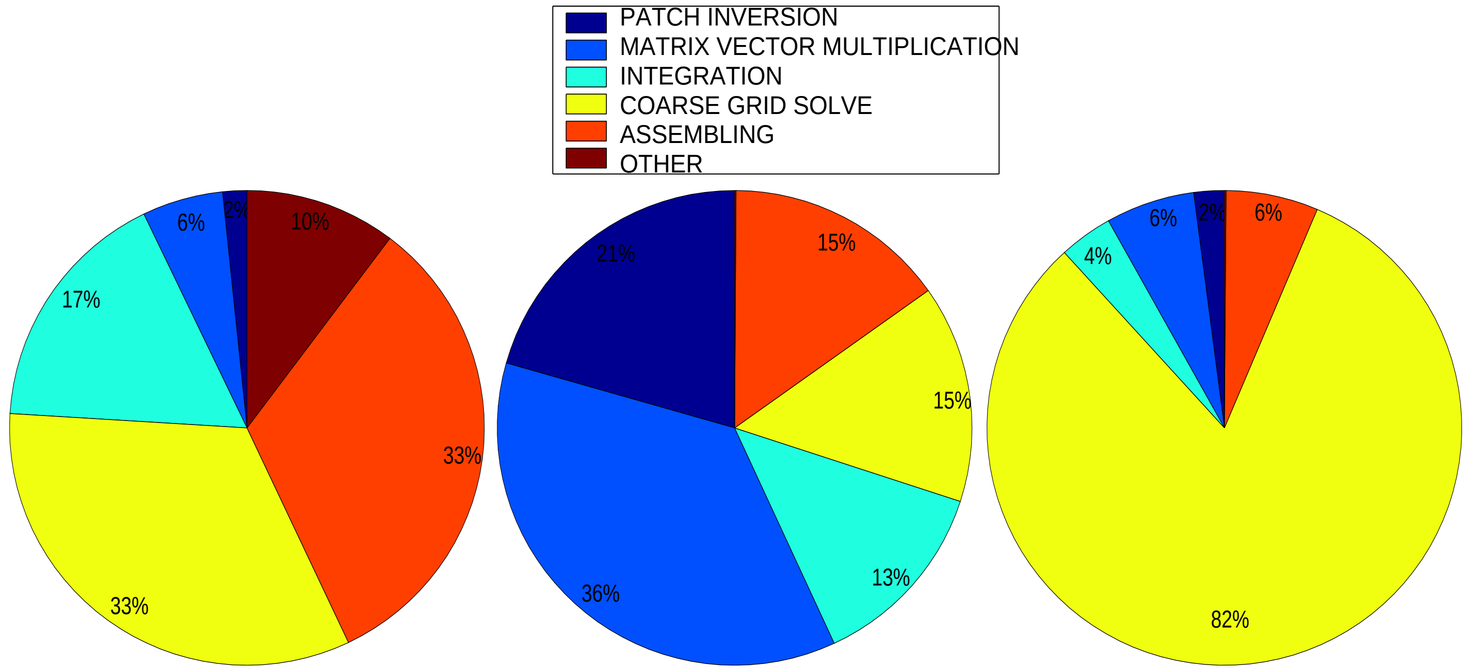


In core computations, AMD Athlon 1 Ghz processor.

7.6 EFFICIENCY

Performance of the two grid solver

3D shock like solution problem



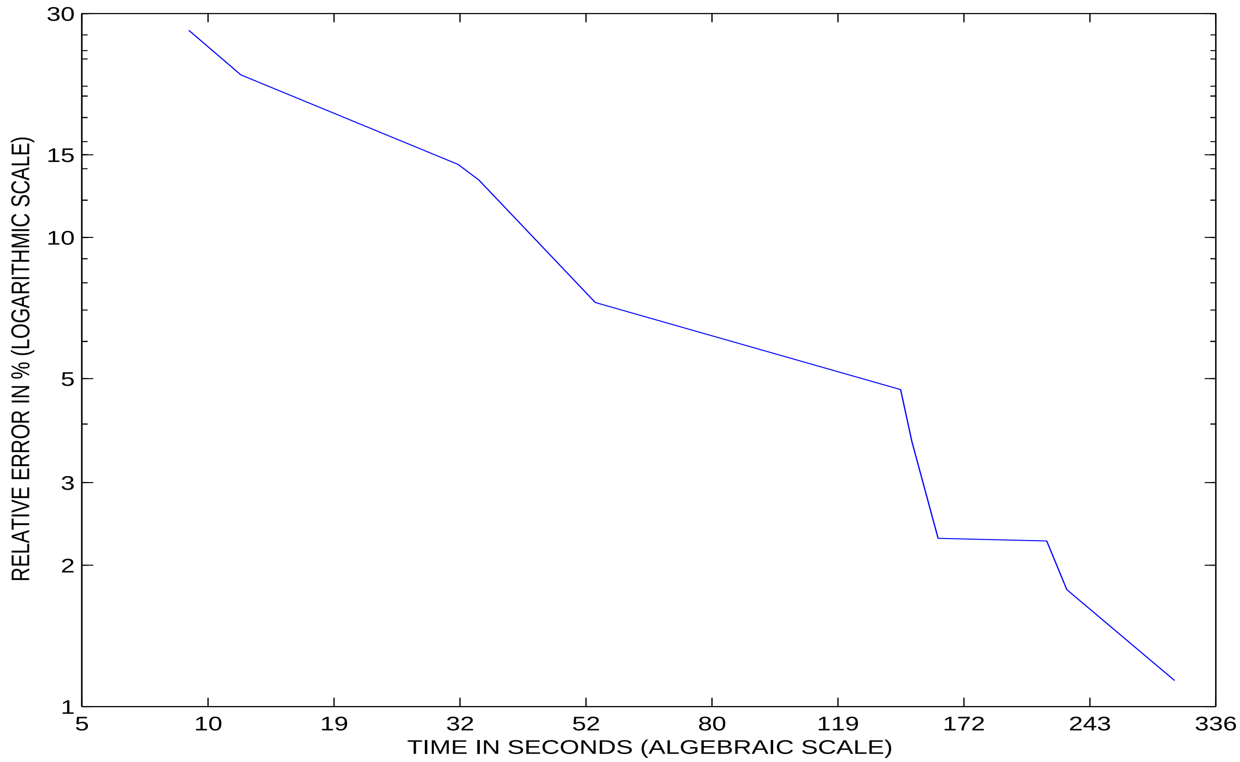
Nrdofs ≈ 2.15 Million	**Nrdofs ≈ 0.27 Million**	**Nrdofs ≈ 2.15 Million**
Total time ≈ 8 minutes	**Total time ≈ 10 minutes**	**Total time ≈ 50 minutes**
Memory* ≈ 1.0 Gb	**Memory* ≈ 2.0 Gb**	**Memory* ≈ 3.5 Gb**
p=2	**p=8**	**p=4**

***Memory = memory used by nonzero entries of stiffness matrix**
In core computations, IBM Power4 1.3 Ghz processor.

7.7 EXPONENTIAL CONVERGENCE

Convergence history

3D shock like solution example.
Scales: ERROR VS TIME.



A TWO GRID SOLVER FOR ELECTROMAGNETICS

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$$r^{(n+1)} = [I - \alpha^{(n)} AS] r^{(n)}$$
$$x^{(n+1)} = x^{(n)} + \alpha^{(n)} S r^{(n)}$$

where S is a matrix, and $\alpha^{(n)}$ is a relaxation parameter. $\alpha^{(n)}$ *optimal* if:

$$\alpha^{(n)} = \arg\ \min \| x^{(n+1)} - x \|_B = \frac{(A^{-1} r^{(n)}, S r^{(n)})_B}{(S r^{(n)}, S r^{(n)})_B}$$ **(NOT COMPUTABLE)**

Then, we define our two grid solver for Electromagnetics as:

1 Iteration with $S = S_F = \sum A_i^{-1} \quad +$

1 Iteration with $S = S_\nabla = \sum G_i^{-1} \quad +$

1 Iteration with $S = S_C = P A_C^{-1} R$

A TWO GRID SOLVER FOR ELECTROMAGNETICS

A two grid solver for discretization of Maxwell's equations using hp-FE

Consider the following two problems:

Problem I: $\nabla \times \nabla \times \mathbf{E} - k^2 \mathbf{E} = \mathbf{J}$

Matrix form: $Au = v$

Two grid solver V-cycle:

$TG = (I - \alpha_1 S_F A)(I - \alpha_2 S_\nabla A)(I - S_C A_C)$

Problem II: $\nabla \times \nabla \times \mathbf{E} + \mathbf{E} = \mathbf{J}$

Matrix form: $\hat{A}u = v$

Two grid solver V-cycle:

$\widehat{TG} = (I - \alpha_1 \hat{S}_F \hat{A})(I - \alpha_2 \hat{S}_\nabla \hat{A})(I - \hat{S}_C \hat{A}_C)$

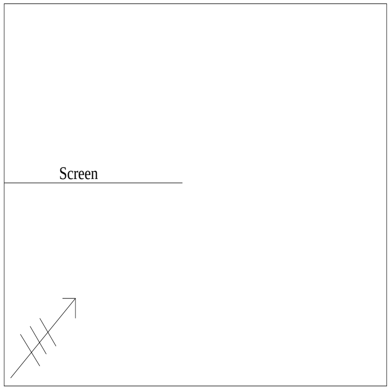
Theorem: If h is small enough, then:

$$\| TGe^{(n)} \| \leq \| \widehat{TG}e^{(n)} \| + Ch$$

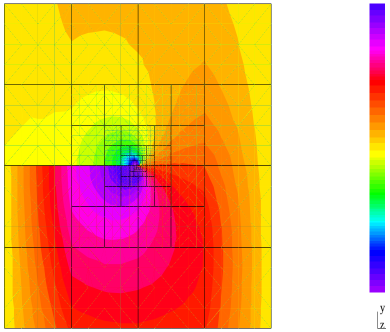
Notice that C is independent of h and p.

A TWO GRID SOLVER FOR ELECTROMAGNETICS

Plane Wave incident into a screen (diffraction problem)

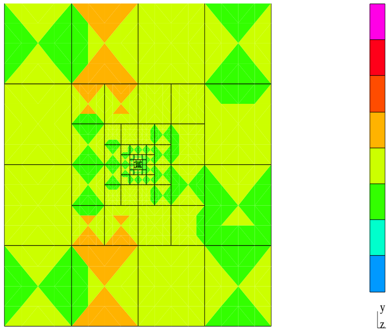


Geometry



Second component of electric field

Convergence history
(tolerance error = 0.1 %)

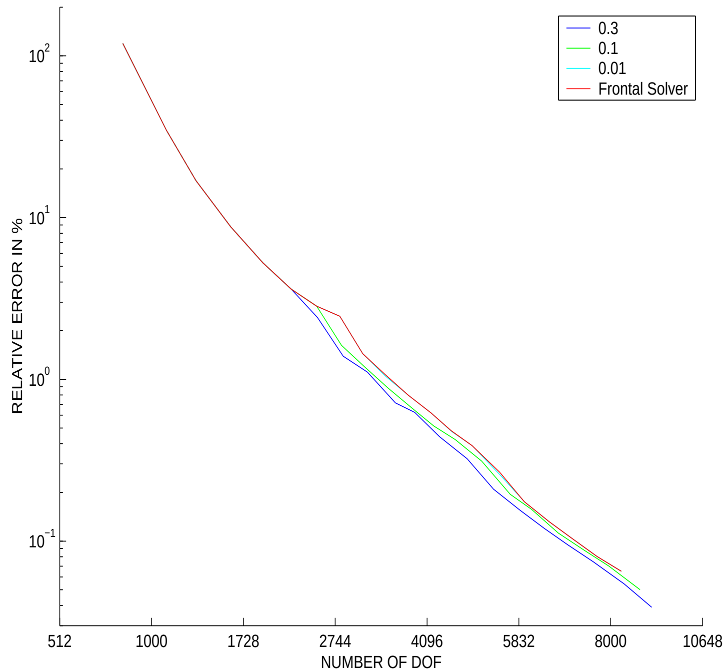


Final hp-grid

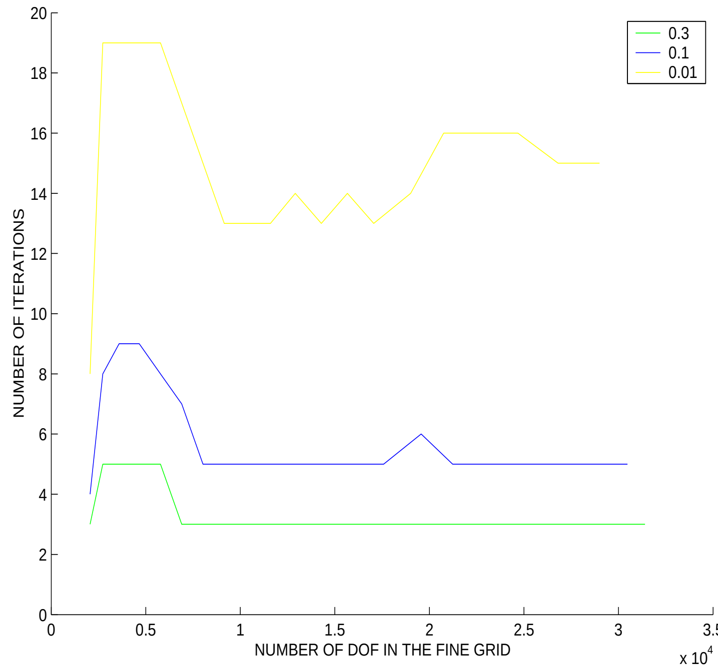
Numerical Results

Guiding automatic hp-refinements

Diffraction problem. Guiding hp-refinements with a partially converged solution.



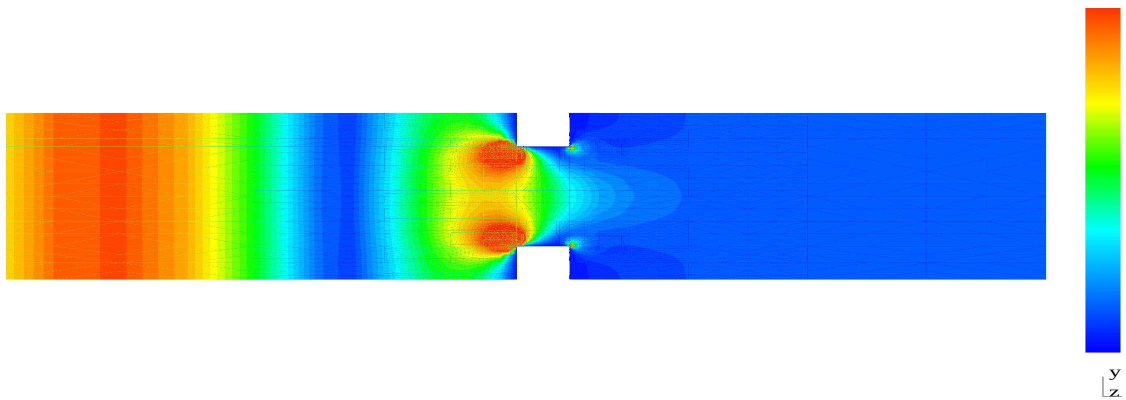
Discretization error estimate



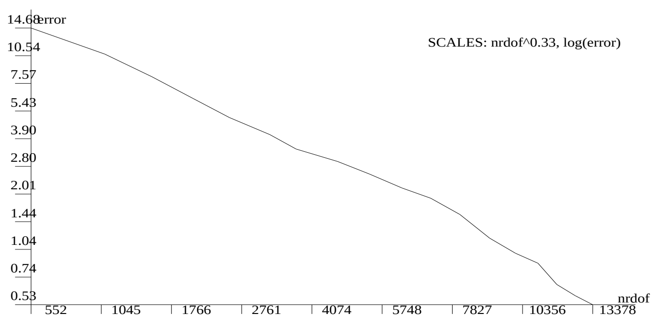
Number of iterations

A TWO GRID SOLVER FOR ELECTROMAGNETICS

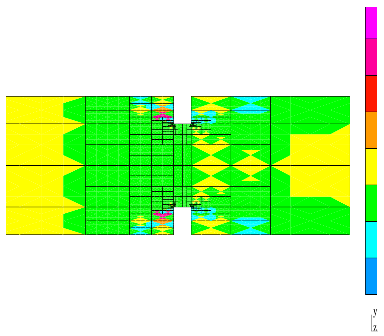
Waveguide example



Module of Second Component of Magnetic Field



Convergence history (tolerance error = 0.5 %)

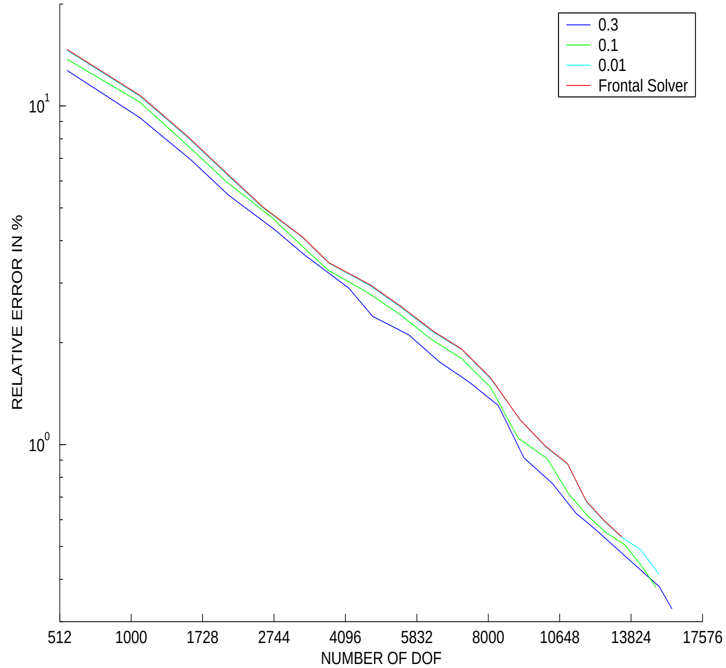


Final hp-grid

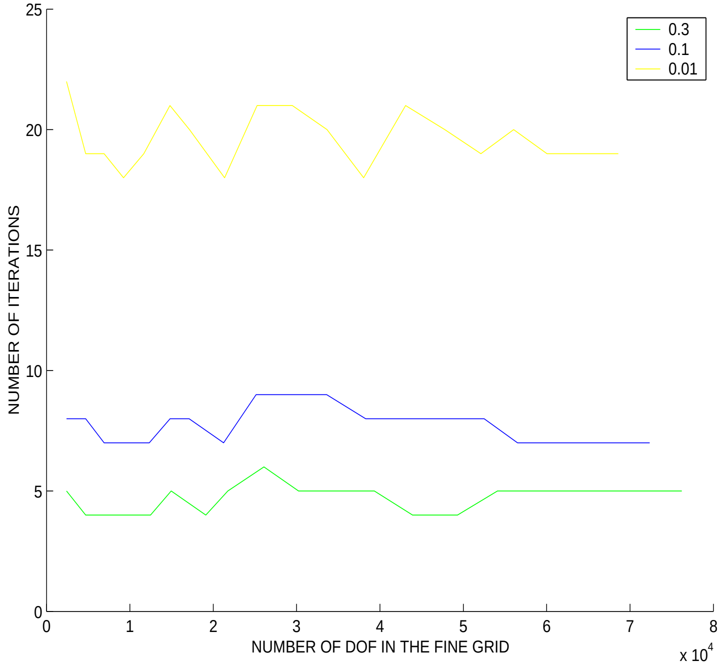
A TWO GRID SOLVER FOR ELECTROMAGNETICS

Guiding automatic hp-refinements

Waveguide example. Guiding hp-refinements with a partially converged solution.



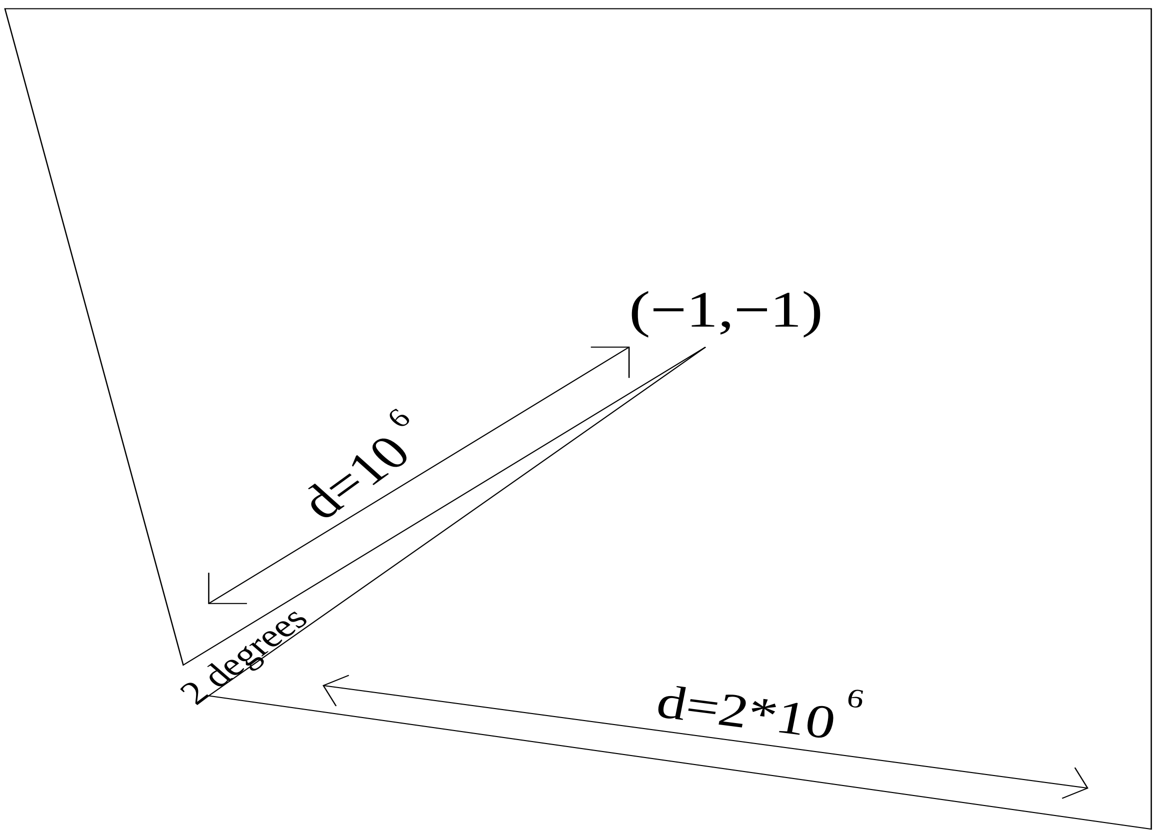
Discretization error estimate



Number of iterations

9. ELECTROMAGNETIC APPLICATIONS

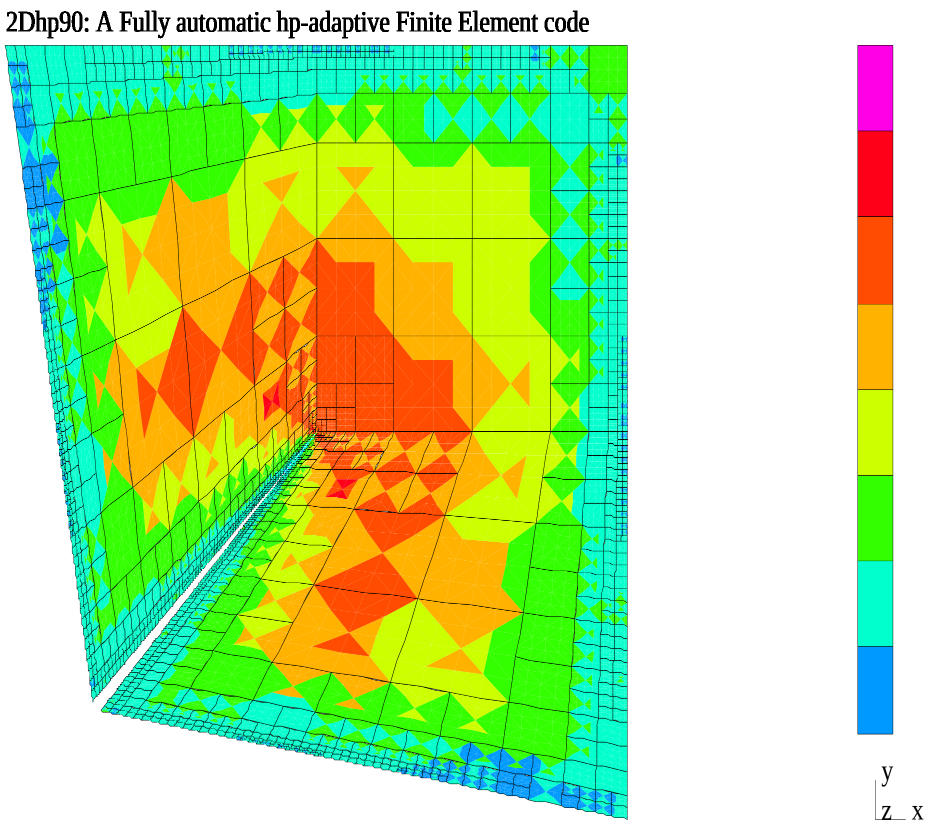
Edge diffraction example (Baker-Hughes): Electrostatics



Dirichlet Boundary Conditions
u(boundary)=−ln r, r=sqrt (x*x+y*y)

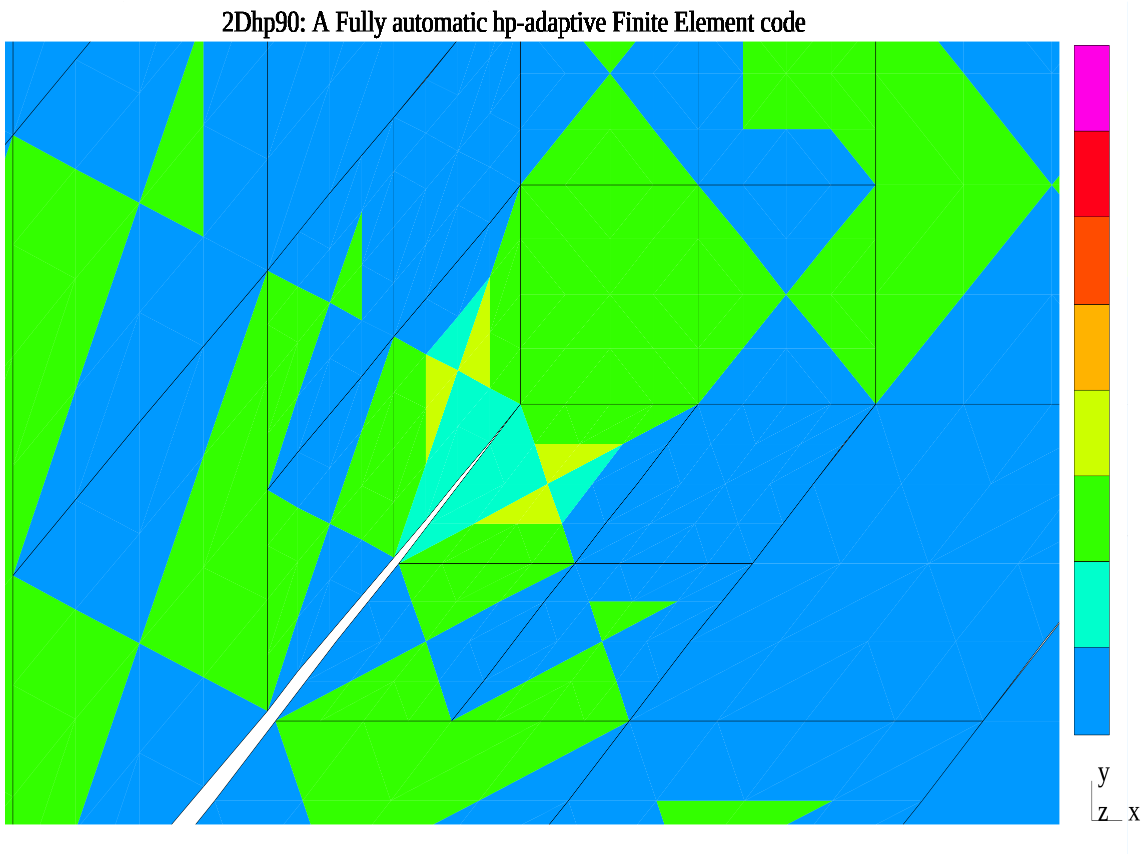
9. ELECTROMAGNETIC APPLICATIONS

Edge diffraction example: final *hp*-grid, Zoom = 1



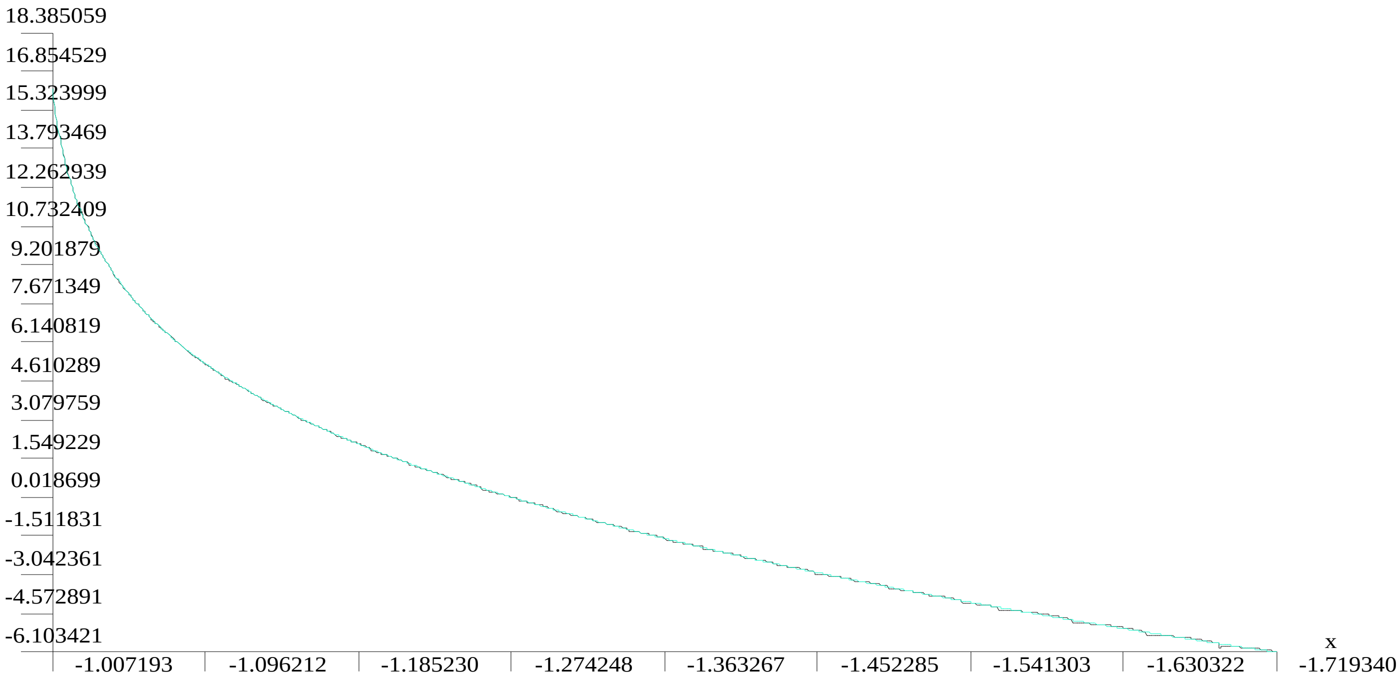
9. ELECTROMAGNETIC APPLICATIONS

Edge diffraction example: final *hp*-grid, Zoom = 10^{13}



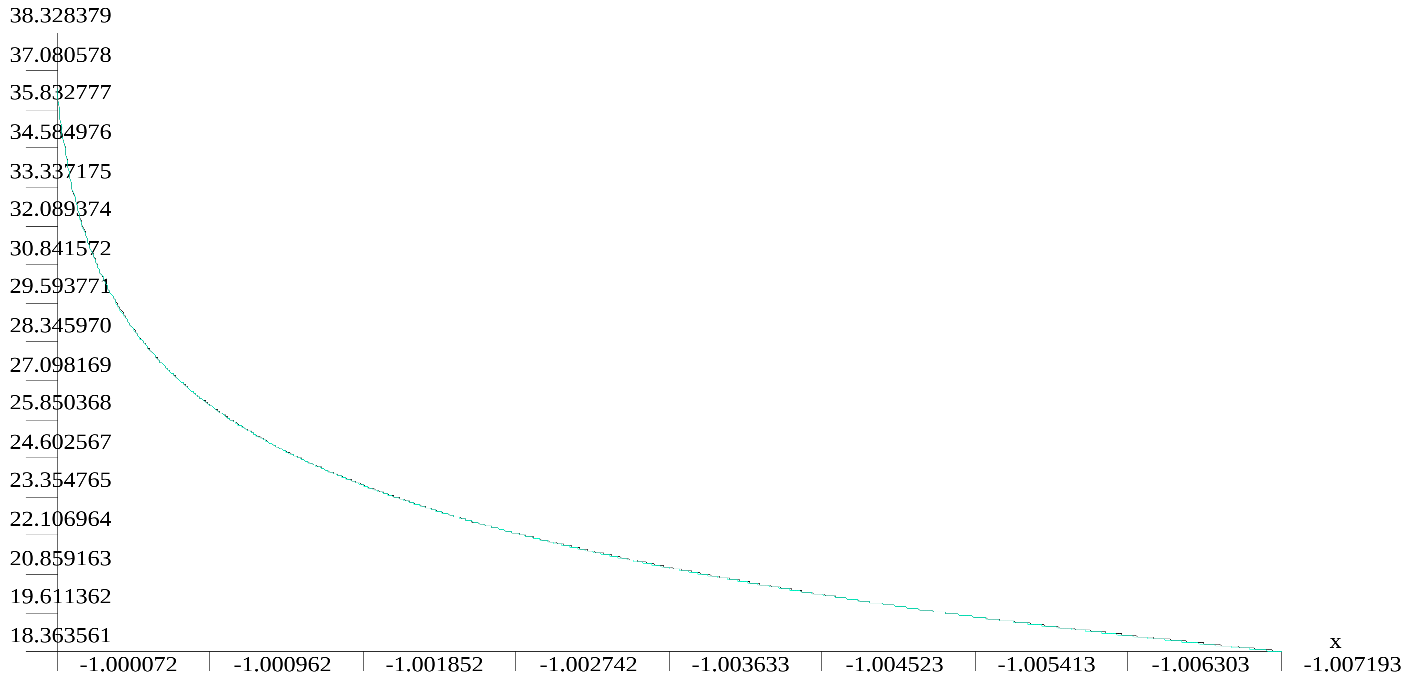
9. ELECTROMAGNETIC APPLICATIONS

Edge diffraction example: Comparison between exact and approximate solution at distances 0.01-1 from the singularity



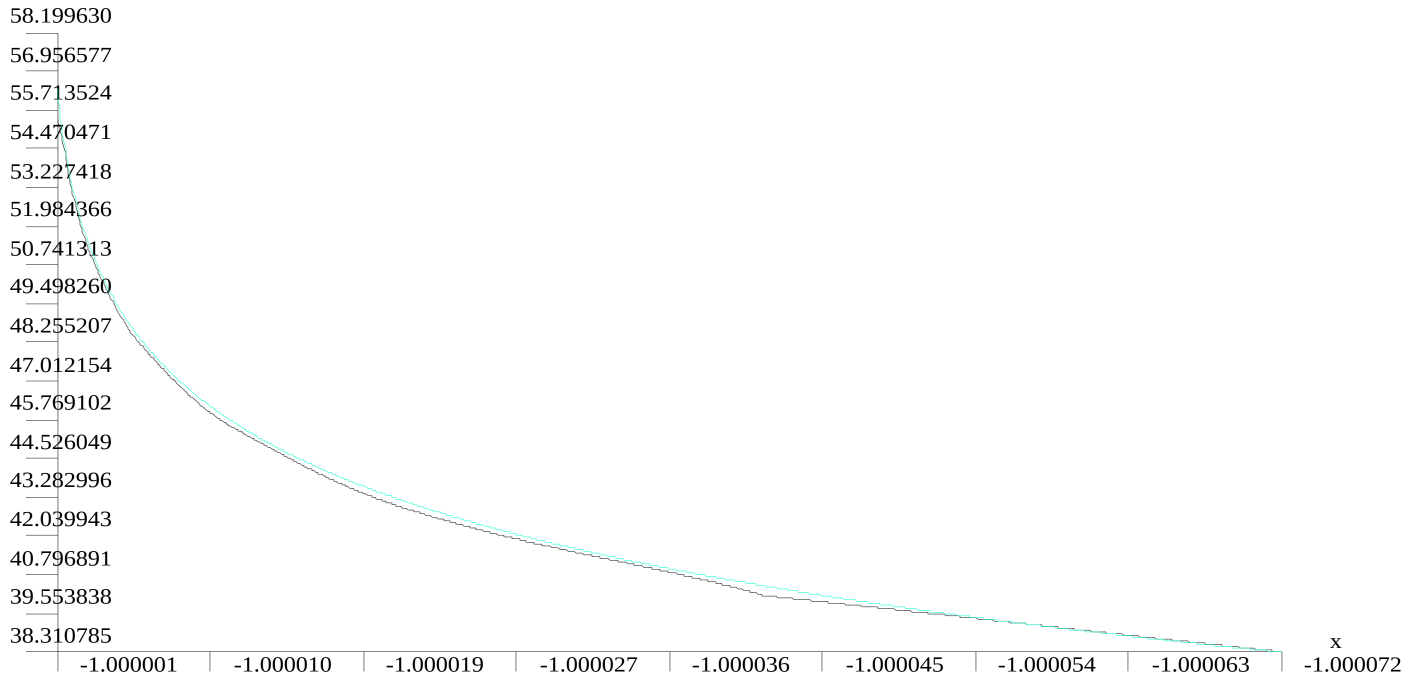
9. ELECTROMAGNETIC APPLICATIONS

Edge diffraction example: Comparison between exact and approximate solution at distances 0.0001-0.01 from the singularity



9. ELECTROMAGNETIC APPLICATIONS

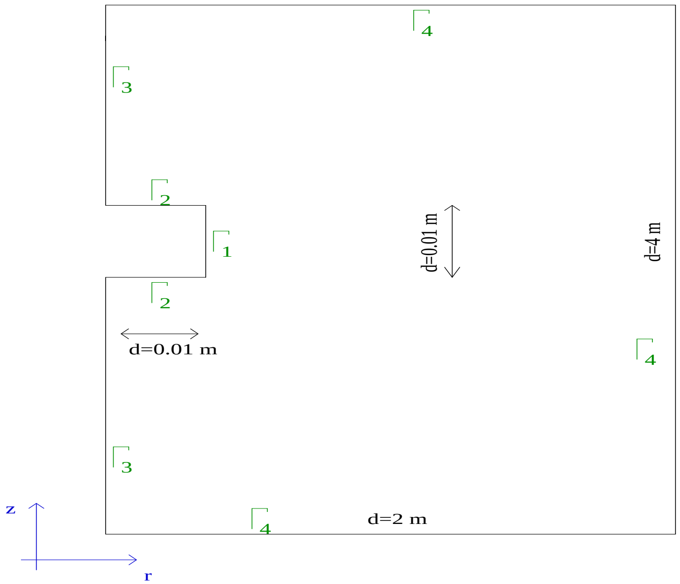
Edge diffraction example: Comparison between exact and approximate solution at distances 0.000001-0.0001 from the singularity



9. ELECTROMAGNETIC APPLICATIONS

Time Harmonic Maxwell's Equations

$$\nabla \times \mathbf{E} = -j\mu\omega\mathbf{H}$$
$$\nabla \times \mathbf{H} = j\omega\epsilon\mathbf{E} + \sigma\mathbf{E}$$



Reduced Wave Equation:

$$\boldsymbol{\nabla} \times \left(\frac{1}{\mu}\boldsymbol{\nabla} \times \boldsymbol{E}\right) - (\omega^2\epsilon - j\omega\sigma)\boldsymbol{E} = -j\omega J^{imp}$$

Boundary Conditions (BC):

Dirichlet BC at a PEC surface:

$$\mathbf{n} \times \boldsymbol{E} = 0 \ \ on \ \ \Gamma_2 \cup \Gamma_4$$

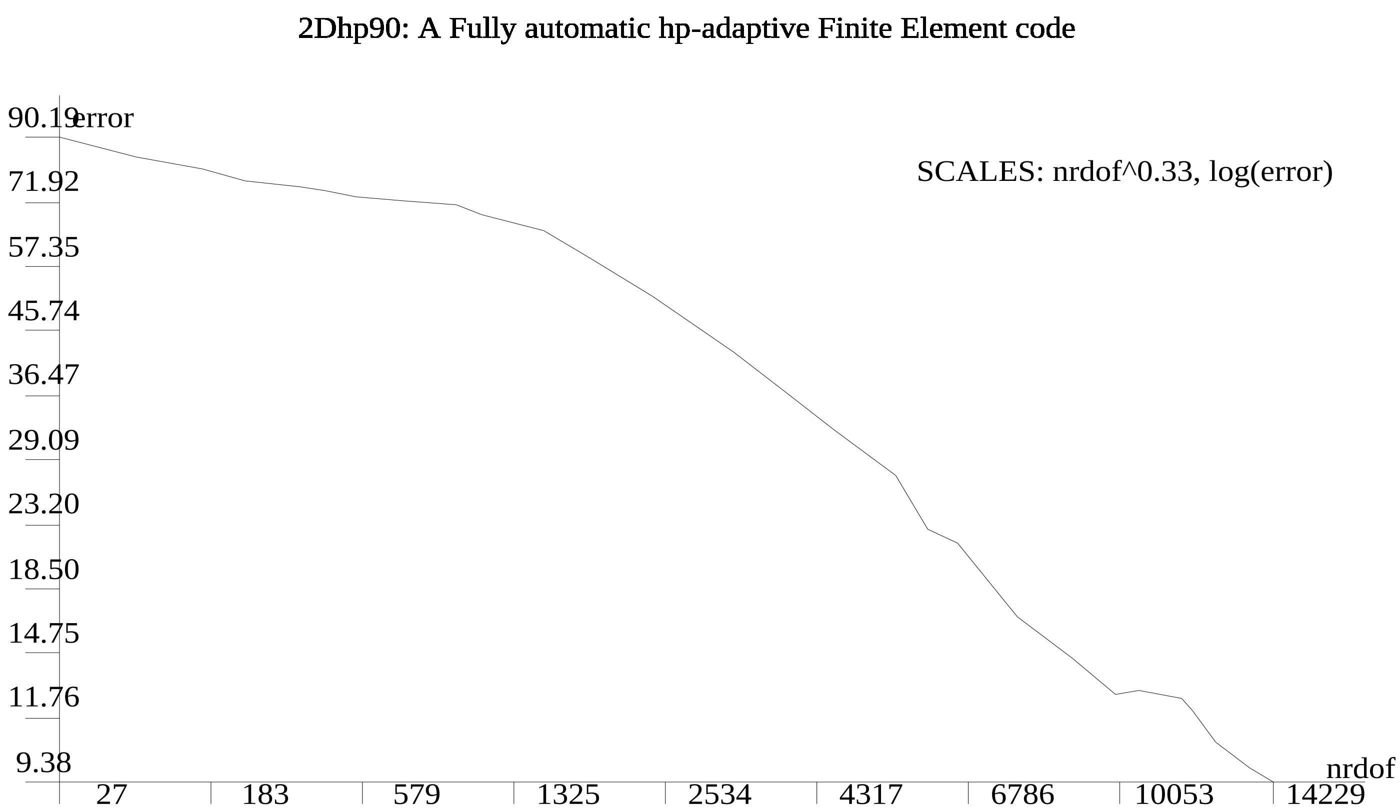
Neumann BC's:

$$\mathbf{n} \times \frac{1}{\mu}\nabla \times \boldsymbol{E} = -j\omega \ \ on \ \ \Gamma_1$$

$$\mathbf{n} \times \frac{1}{\mu}\nabla \times \boldsymbol{E} = 0 \ \ on \ \ \Gamma_3$$

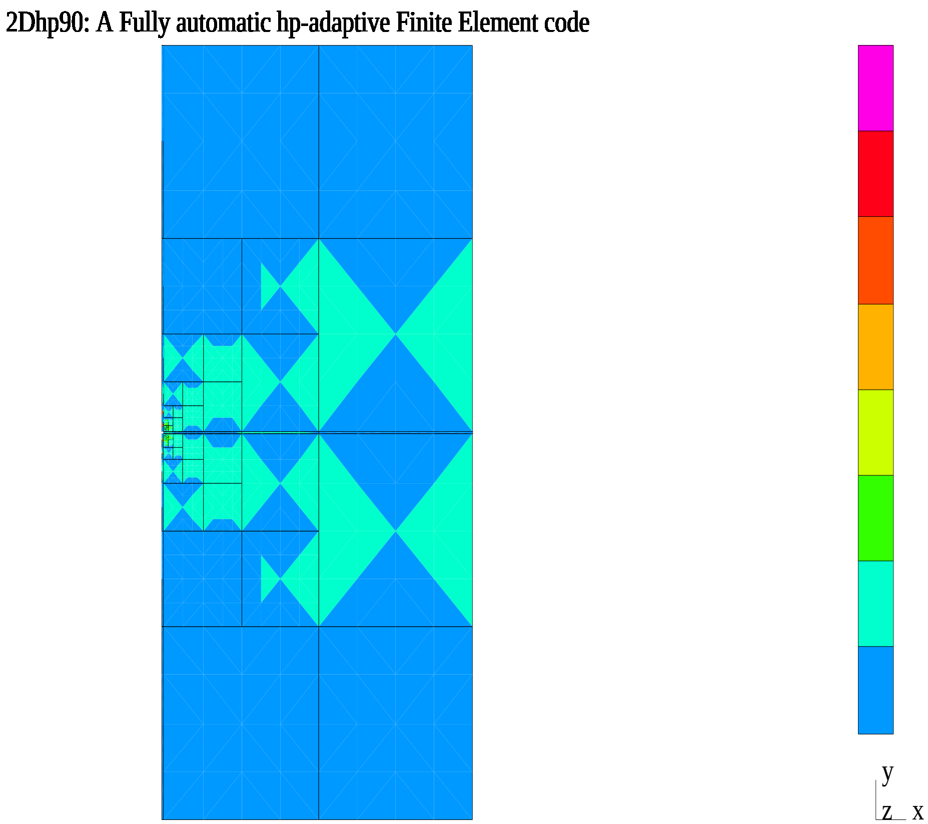
9. ELECTROMAGNETIC APPLICATIONS

Battery example: Convergence history



9. ELECTROMAGNETIC APPLICATIONS

Battery example: final *hp*-grid, Zoom = 1



9. ELECTROMAGNETIC APPLICATIONS

Why the optimal grid is so bad?

Optimization is based on minimization of the ENERGY NORM of the error, given by:

$$\| error \|^2 = \int | error |^2 + \int | \nabla \times error |^2$$

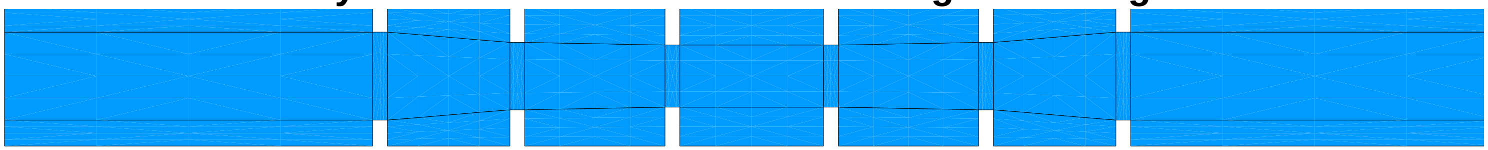
Interpretation of results:

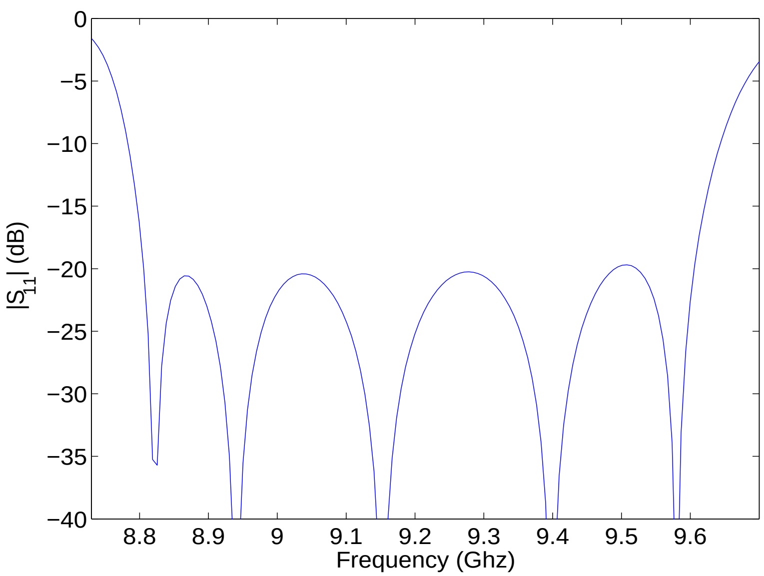
- **The grid is optimal for the selected refinement criteria,**
- **but our refinement criteria is inadequate for our pourposes.**

9. ELECTROMAGNETIC APPLICATIONS

Waveguide example with five iris

Geometry of a cross section of the rectangular waveguide





Return loss of the waveguide structure

H-plane five resonant iris filter.

Dominant mode (source): $TE_{10}-$mode.

Dimensions $\approx 20 \times 2 \times 1$ cm.

Operating Frequency $\approx 8.8 - 9.6$ Ghz

Cutoff frequency ≈ 6.56 Ghz

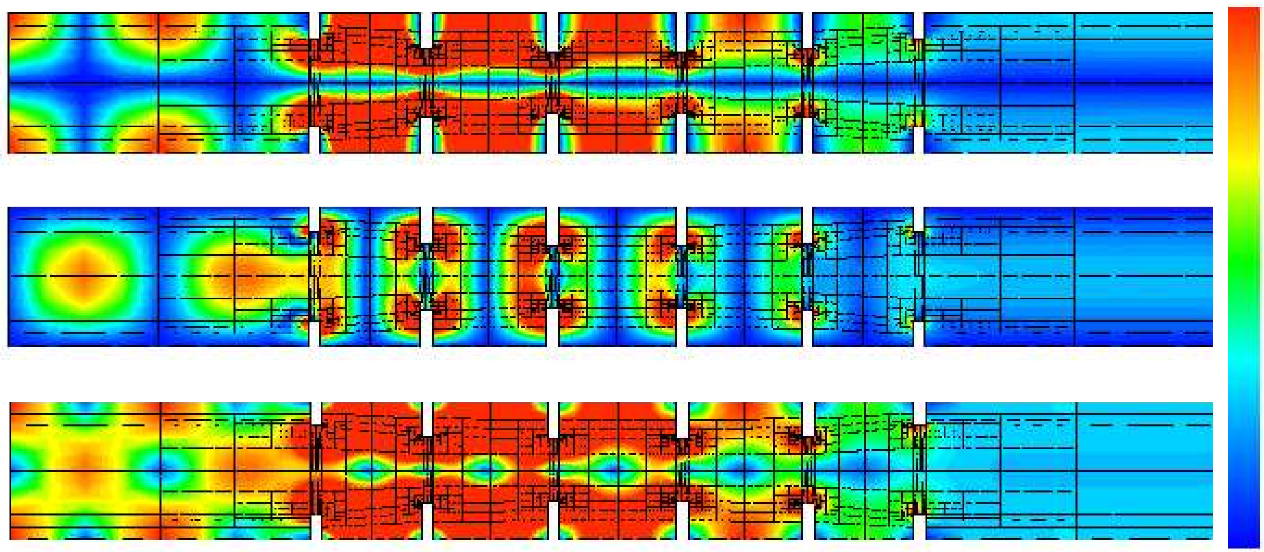
9. ELECTROMAGNETIC APPLICATIONS

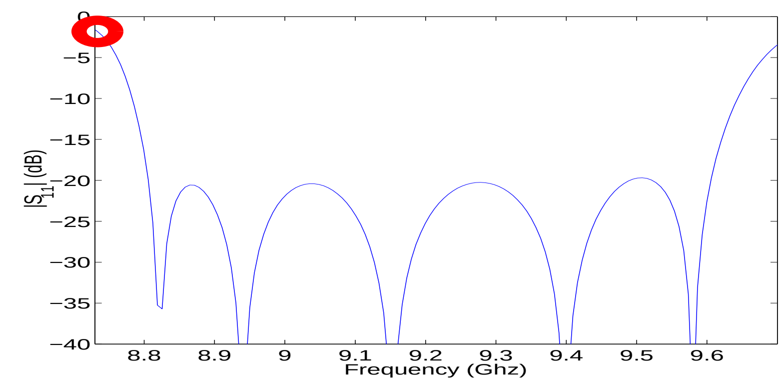
FEM solution for frequency $= 8.72$ Ghz

$|H_x|$

$|H_y|$

$\sqrt{|H_x|^2+|H_y|^2}$





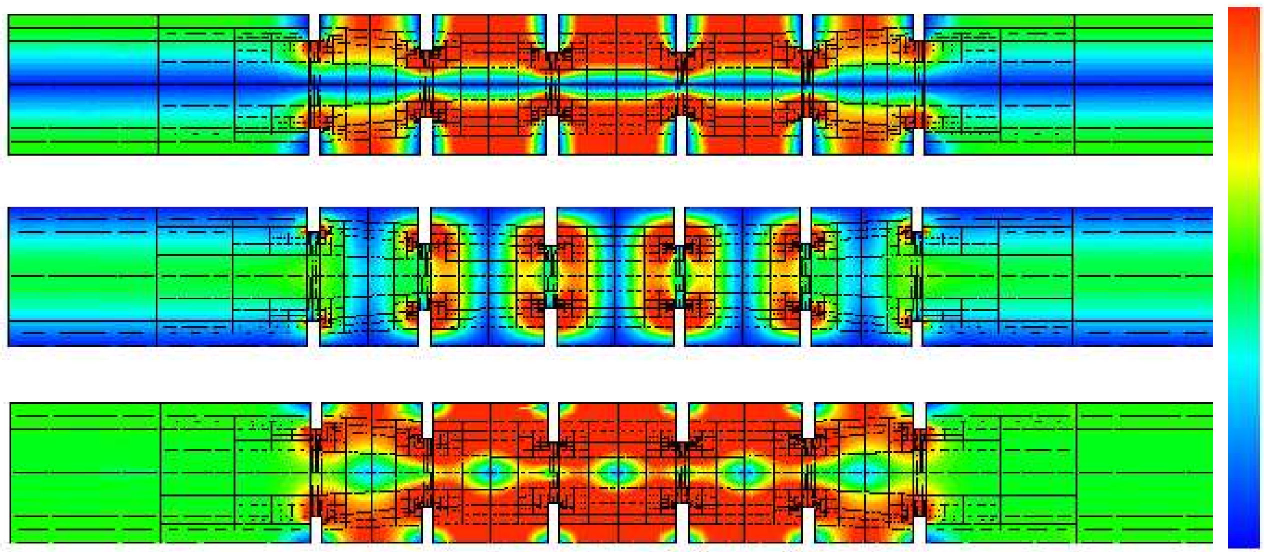
9. ELECTROMAGNETIC APPLICATIONS

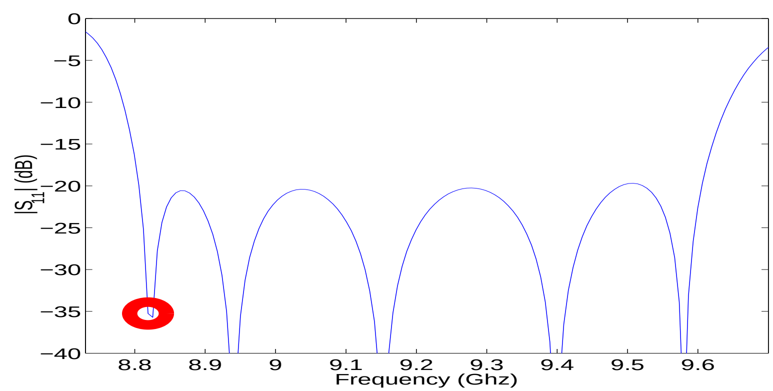
FEM solution for frequency $= 8.82$ Ghz

$|H_x|$

$|H_y|$

$\sqrt{|H_x|^2+|H_y|^2}$





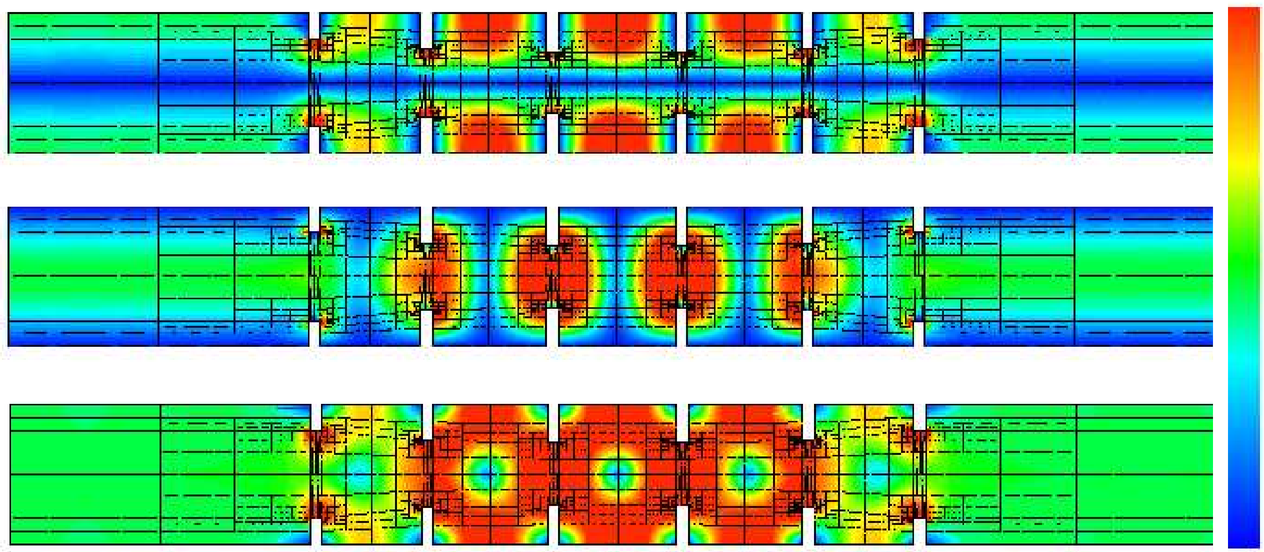
9. ELECTROMAGNETIC APPLICATIONS

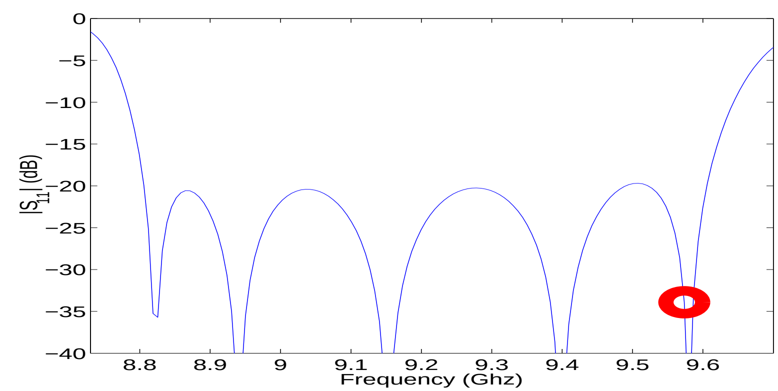
FEM solution for frequency $= 9.58$ Ghz

$|H_x|$

$|H_y|$

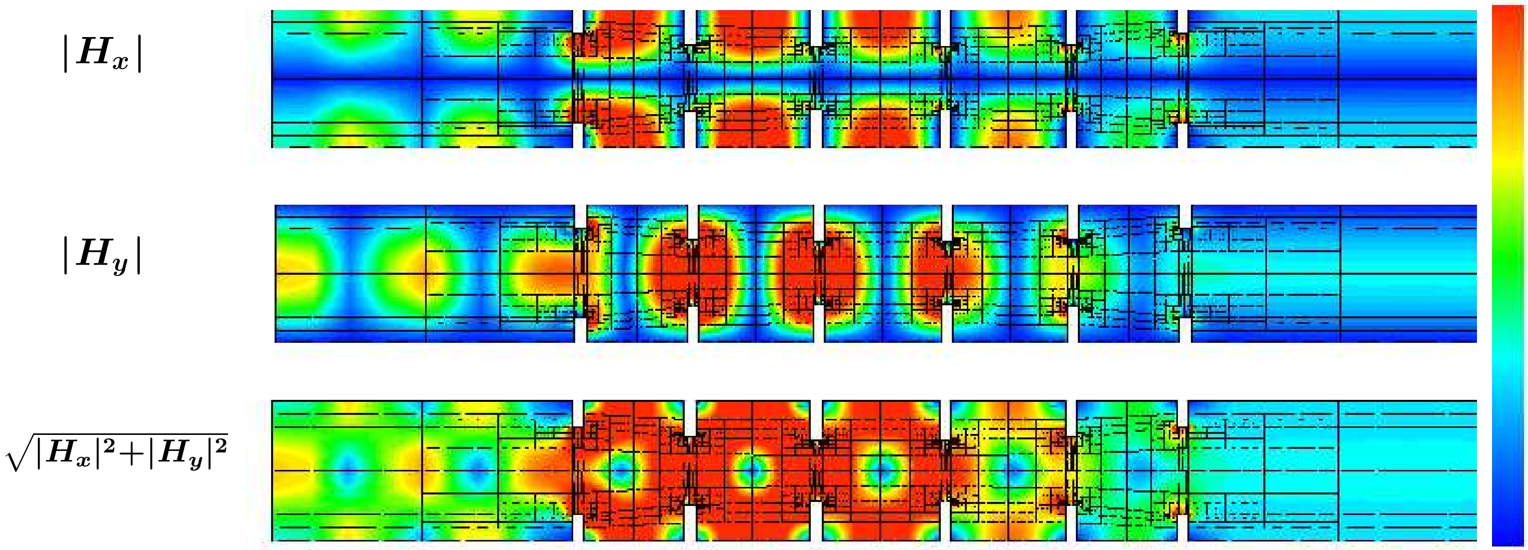
$\sqrt{|H_x|^2+|H_y|^2}$

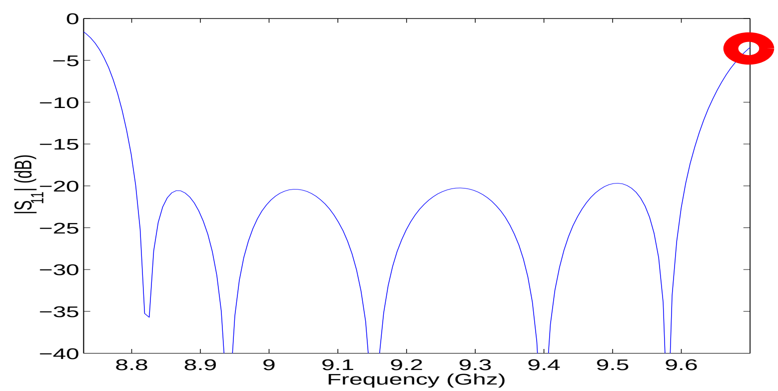




9. ELECTROMAGNETIC APPLICATIONS

FEM solution for frequency $= 9.71$ Ghz



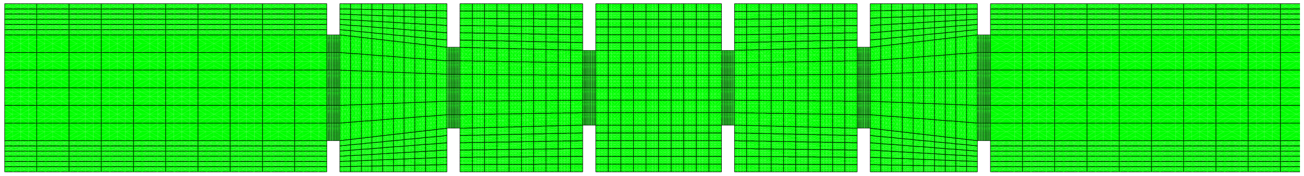


9. ELECTROMAGNETIC APPLICATIONS

Griding Techniques for the Waveguide Problem

Our refinement technology incorporates:

An hp-adaptive algorithm	A two grid solver
Low dispersion error	Convergence of iterative solver
Small h is not enough	Insensitive to p-enrichment $(1 \leq p \leq 4)$
Large p required	Coarse grid sufficiently fine
Waveguide example: $p \approx 3$	Waveguide example: $\lambda/h \approx 9$



Limitations of the hp-strategy for wave propagation problems:

We need large p and small h.

9. ELECTROMAGNETIC APPLICATIONS

Griding Techniques for the Waveguide Problem

Does convergence (or not) of the two grid solver depends upon h and/or p? How?

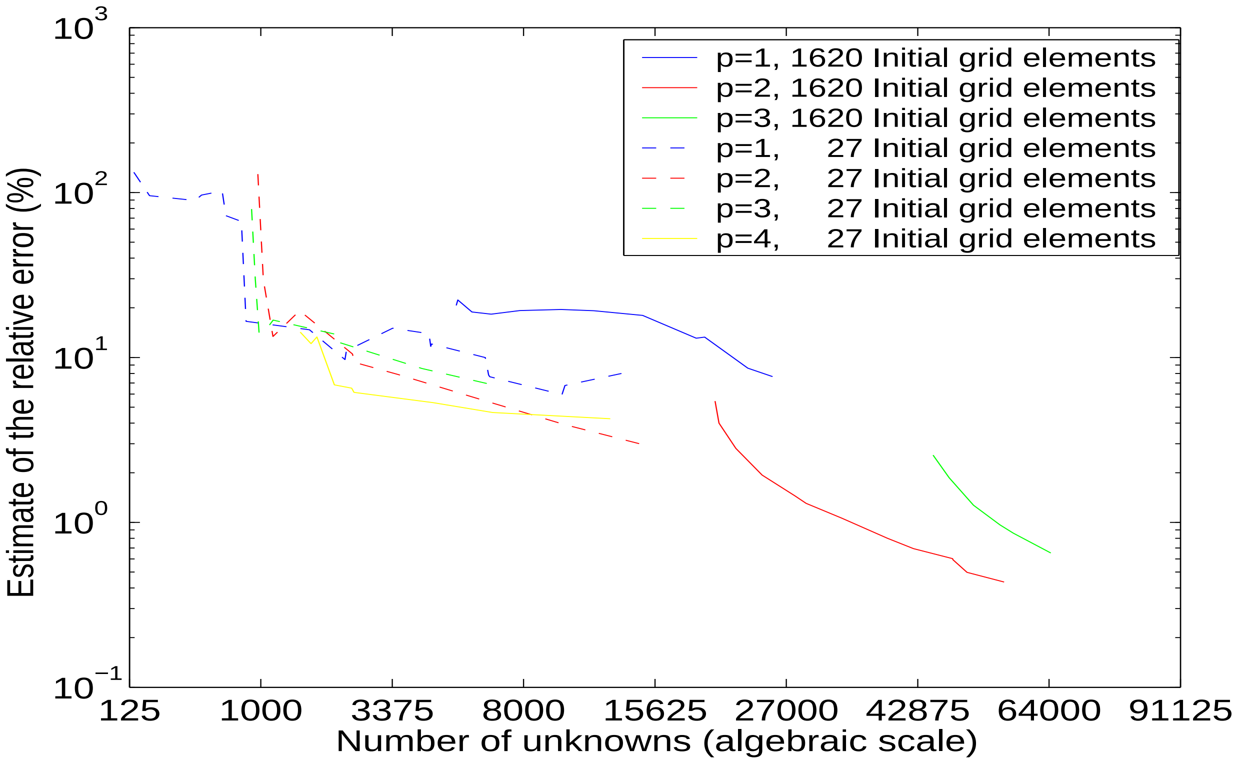
Convergence of two grid solver	$p = 1$	$p = 2$	$p = 3$	$p = 4$
Nr. of elements per $\lambda = 7, 13$	YES	YES	YES	YES
Nr. of elements per $\lambda = 7, 11$	NO	NO	NO	YES
Nr. of elements per $\lambda = 6, 13$	NO	NO	NO	NO

Convergence (or not) of the two grid solver is (almost) insensitive to p-enrichment.

9. ELECTROMAGNETIC APPLICATIONS

Griding Techniques for the Waveguide Problem

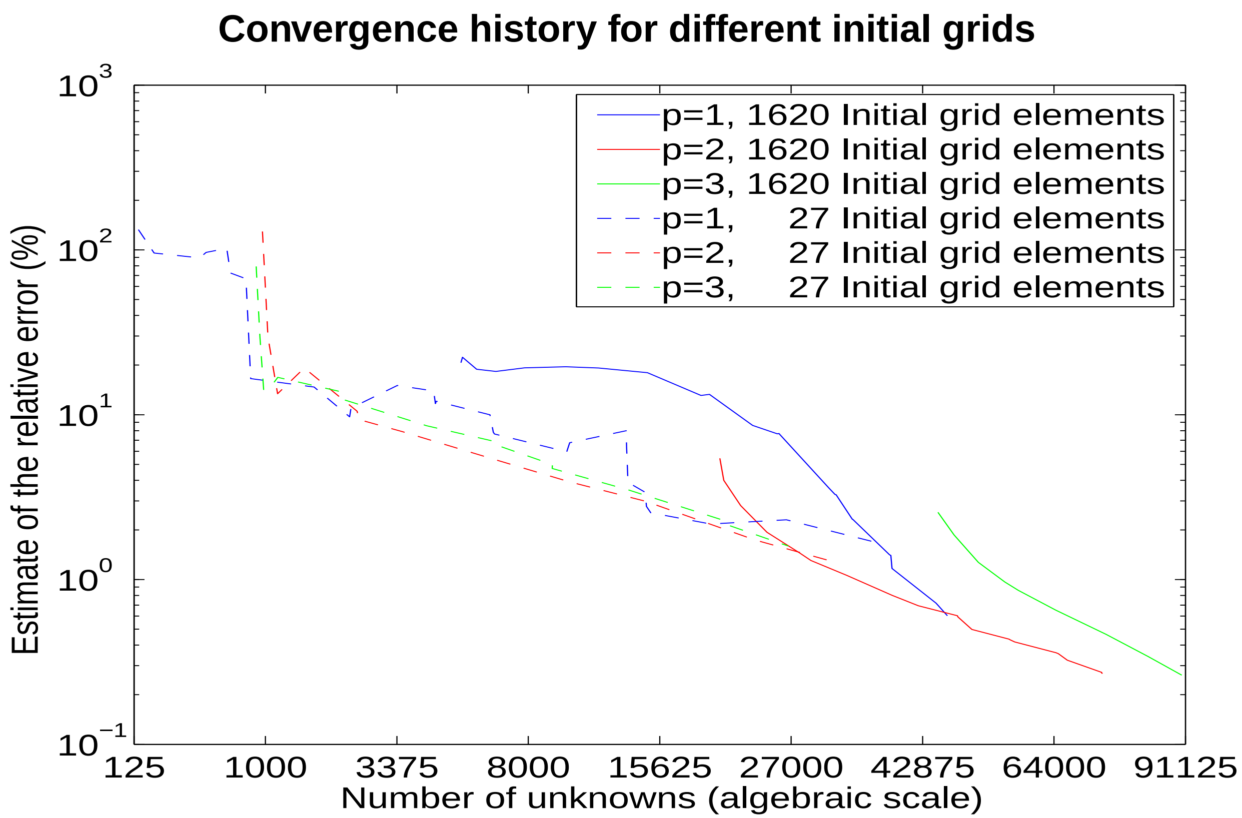
Convergence history for different initial grids



Conclusion : We need to control the dispersion error.

9. ELECTROMAGNETIC APPLICATIONS

Griding Techniques for the Waveguide Problem



Conclusion : **Do we need to control the dispersion error?**

10. CONCLUSIONS AND FUTURE WORK

- **Exponential convergence is achieved for real world problems by using a fully automatic hp-adaptive strategy.**
- **Multigrid for highly nonuniform hp-adaptive grids is an efficient iterative solver.**
- **It is possible to guide hp-adaptivity with partially converged solutions.**
- **This numerical method can be applied to a variety of real world EM problems.**
- **A number of real world EM problems require goal-oriented hp-adaptivity.**