A GEOMETRIC DESCRIPTION OF THE FORMATION OF SINGULARITIES IN THE BINORMAL FLOW

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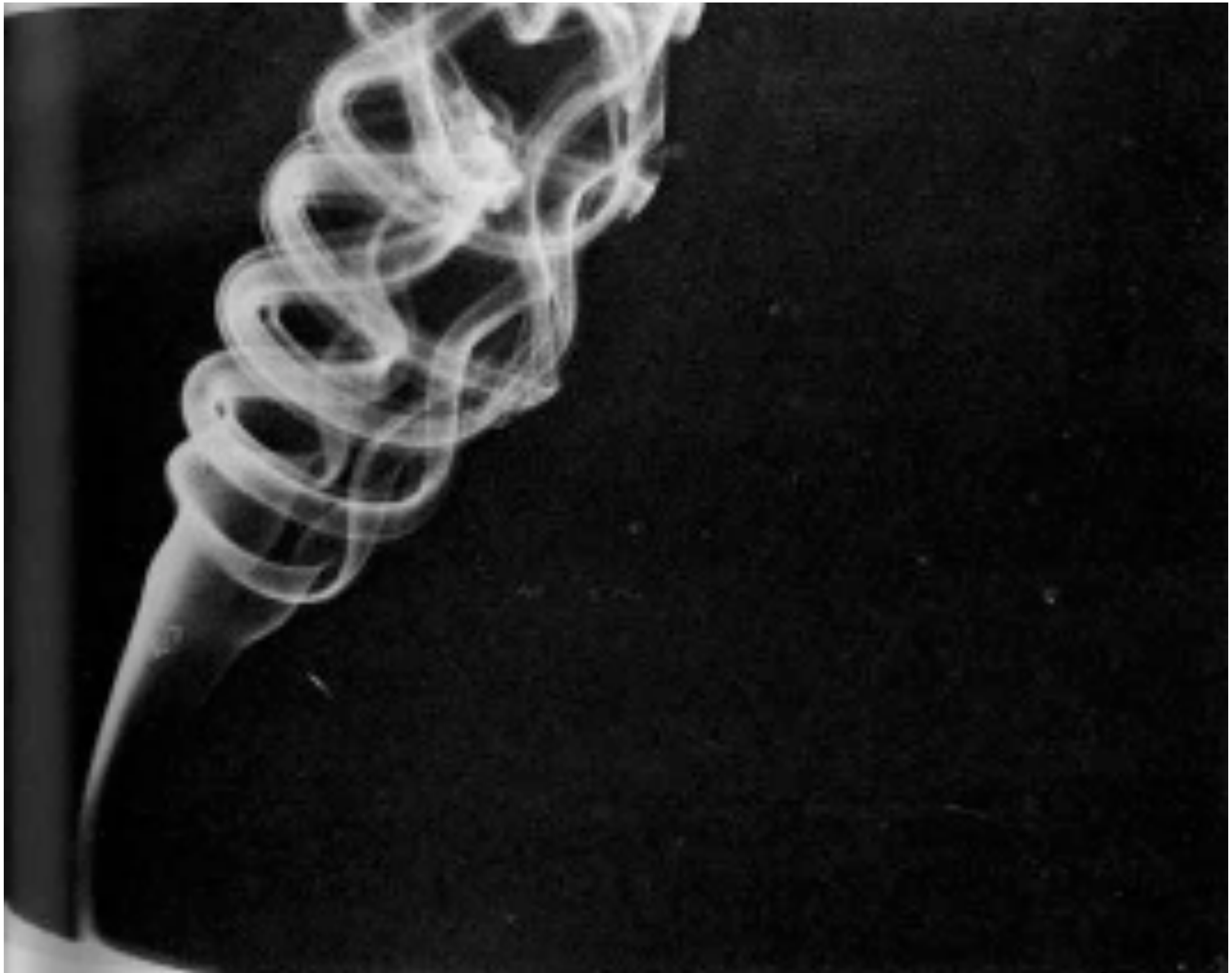
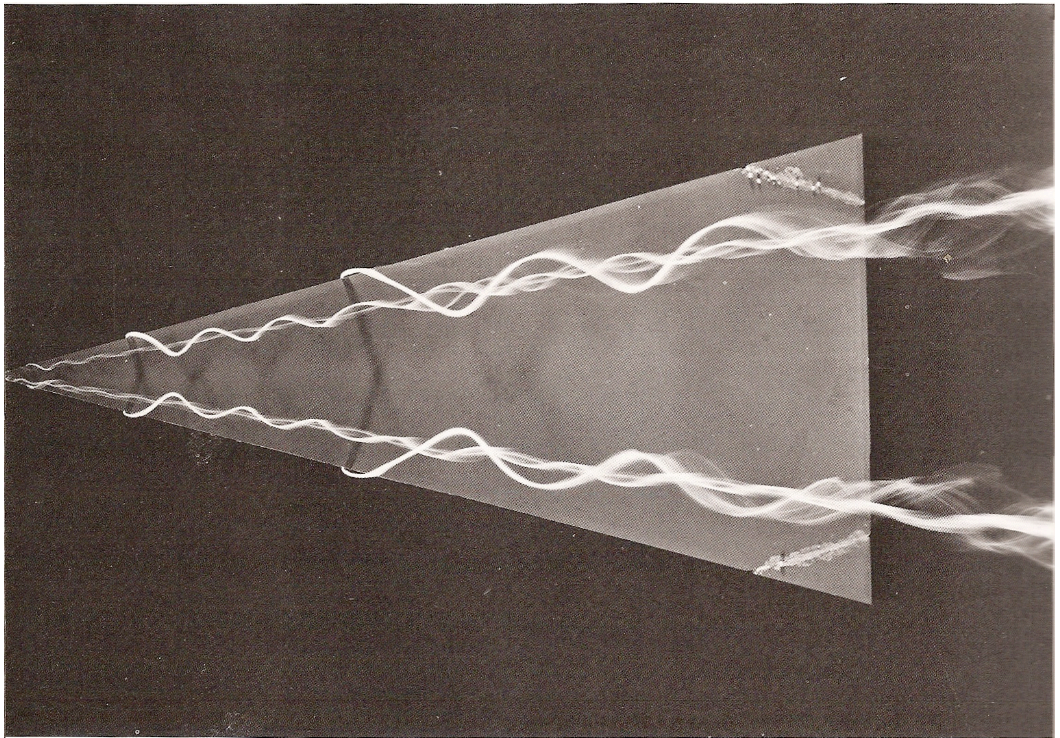
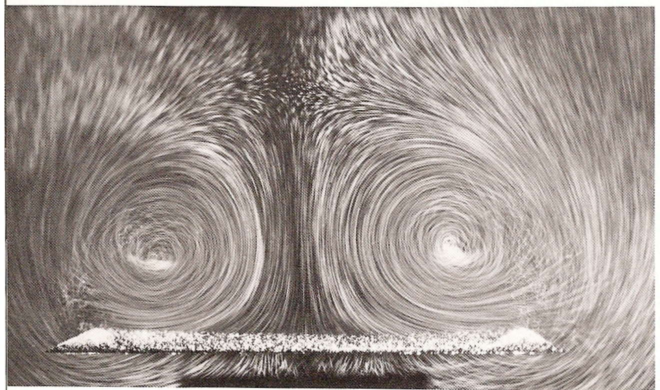




90. Vortices above an inclined triangular wing. Lines of colored fluid in water show the symmetrical pair of vortices behind a thin wing of 15° semi-vertex angle at 20° angle of attack. The Reynolds number is 20,000 based on chord. Although the Mach number is very low, the flow field is practically conical over most of the wing, quantities being constant along rays from the apex. *ONERA photograph, Werlé 1963*



91. Cross section of vortices on a triangular wing. Tiny air bubbles in water show the vortex pair for the flow above in a section at the trailing edge of the wing. *ONERA photograph, Werlé 1963*

Euler equations

u: velocity field

$\omega = \mathrm{curl}\, u = \nabla \wedge u :$ vorticity

$\omega = \Gamma T ds \qquad T = X_s$

$X = X(t,s)$ curve in $\mathbb{R}^3$ support of ω

$\mathrm{div}\, u = 0$

$$u(P) = \frac{\Gamma}{4\pi} \int_{-\infty}^{\infty} \frac{X(s) - P}{|X(s) - P|^3} \wedge T(s) ds$$

Examples: straight lines, vortex rings, helical vortices

BINORMAL FLOW (Vortex Filament Flow)

$$X_t = X_s \wedge X_{ss} = cb$$

- $X = X(t, s) \in \mathbb{R}^3$
- $c = c(t, s)$ curvature
- $b = b(t, s)$ binormal

Examples:

a) circle

b) straight line

c) helix

Remark.– $X_s = T$ $|T|^2 = \text{constant}$

SELF SIMILAR solutions

$$X(t,s) = \sqrt{t}G\left(s/\sqrt{t}\right) \qquad T(t,s) = T\left(s/\sqrt{t}\right)$$

$$T_t = T \wedge T_{ss}$$

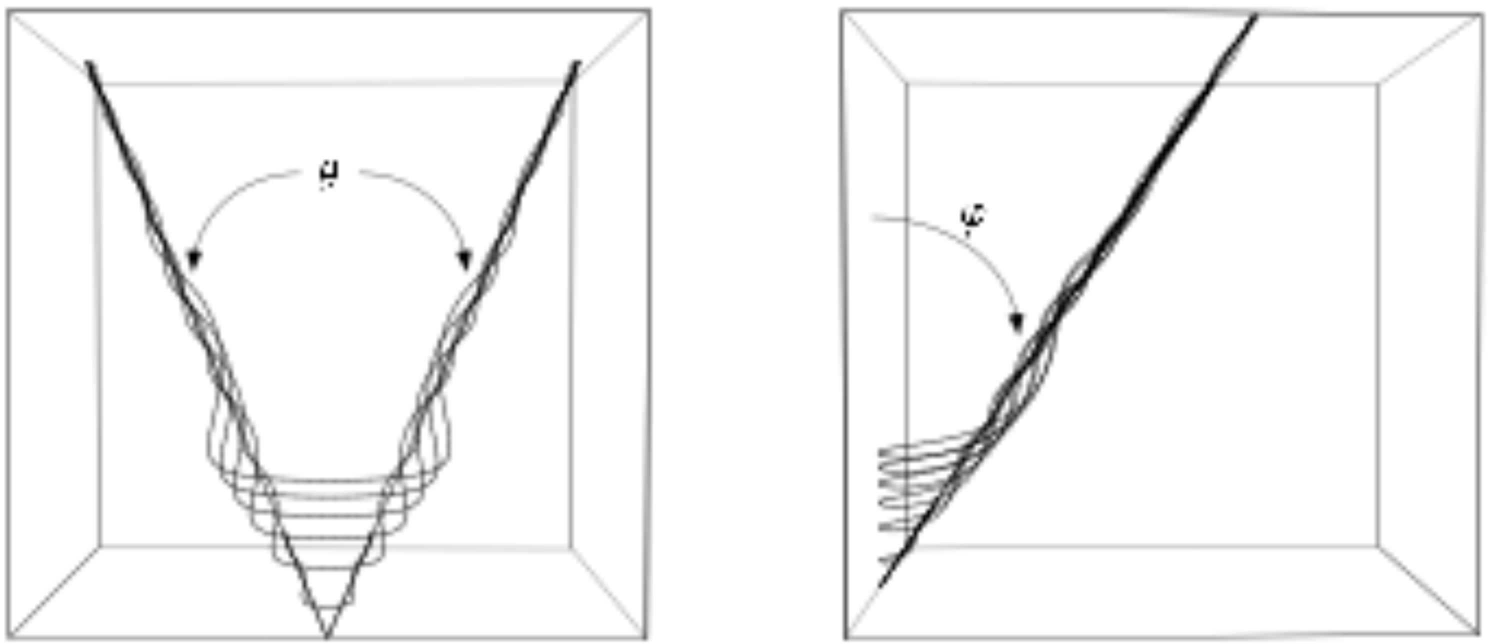
Differentiating and making $t = 1$

$$-\frac{s}{2}T' = T \wedge T_{ss}$$

Frenet equations:

$$\begin{aligned}
T' &= cn \\
n' &= -cT + \tau b \\
b' &= -\tau n \\
-\frac{s}{2}cn &= T \wedge \left(c'n - c^2T + c\tau b\right) \\
c' &= 0 \qquad c = a \qquad \tau = s/2
\end{aligned}$$

Buttke'88



$$G'(s) = T \qquad \frac{1}{2}G - \frac{s}{2}G' = ab$$

$$\left(\frac{G}{s}\right)' = \frac{G's - G}{s^2} = -2a\frac{b}{s^2} \qquad |b| = 1$$

$$\frac{G}{s} \longrightarrow A^{\pm} \qquad s \longrightarrow \pm\infty$$

- $X(1) = G$ is regular
- $X(t,s) = \sqrt{t}G\left(s/\sqrt{t}\right)$: $\qquad X(0,s) = \begin{cases} A^+ s & s > 0 \\ A^- s & s < 0 \end{cases}$
- **Q** : $A^+ = A^-$?

NO $\qquad \displaystyle\int_{-\infty}^{+\infty} e^{is^2 + ia \lg s} \frac{ds}{s}.$

$A^+ = A^-$? $\qquad \theta = \text{angle}\left(A^+, A^-\right)$

$G \qquad c = a \qquad \tau = s/2$

Lemma.– Let c and τ be the curvature and the torsion of a curve in $\mathbb{R}^3$ with tangent $T = (T_1, T_2, T_3)$. Then

$$T_j = 1 - |\eta_j|^2 \qquad j = 1, 2, 3$$

for some η_j solution of

$$\eta_j'' + \left(i\tau - \frac{c'}{c}\right)\eta_j' + \frac{c^2}{4}\eta_j = 0$$

In our case

- $\eta'' + i\frac{s}{2}\eta' + \frac{a^2}{4}\eta = 0$

- $\eta(0) = 0 \qquad , \qquad \eta'(0) = \frac{c_0}{\sqrt{2}}$

- $\lim_{s\to\infty} 1 - |\eta(s)|^2 = \sin\frac{\theta}{2} = e^{-\pi\frac{a^2}{2}}$

$$\beta(s) = \int e^{is\xi + i\xi^2}|\xi|^{-i\frac{a^2}{2}}\frac{d\xi}{\xi}.$$

(Joint work with Gutierrez and Rivas)

There exist $A_a^{\pm}$, $E_a^{\pm}$

- $\sqrt{t}G_a\left(s/\sqrt{t}\right) = A_a^{\pm}s + \mathcal{O}\left(\frac{\sqrt{t}}{s}\right)$
- $T_a\left(s/\sqrt{t}\right) = A_a^{\pm} + \mathcal{O}\left(\frac{\sqrt{t}}{s}\right)$
- $(n_a - ib_a)\left(s/\sqrt{t}\right) = E_a^{\pm}e^{is^2/4t + ia^2 \lg \frac{|s|}{\sqrt{t}}} + \mathcal{O}\left(\frac{\sqrt{t}}{s}\right)$

SCHRÖDINGER EQUATION

Hasimoto transformation:

$$\psi(t,s) = c(t,s)e^{i\int_0^s \tau(t,s')ds'}$$

$c = c(t,s)$ curvature

$\tau = \tau(t,s)$ torsion

$$\partial_t \psi(t,s) = i\left(\partial_s^2 \psi \pm \frac{1}{2}(|\psi|^2 + A(t))\psi\right)$$

$$\int_{-\infty}^{\infty} |\psi(t,s)|^2 ds = \int_{-\infty}^{\infty} |\psi(0,s)|^2 ds = \int_{-\infty}^{\infty} c^2(0,s) ds$$

In our case

$$\psi(t,s) = \frac{a}{\sqrt{t}} e^{i\frac{s^2}{4t}} \qquad , \qquad \int_{-\infty}^{\infty} |\psi|^2 ds = +\infty.$$

STABILITY / INSTABILITY (V. Banica)

The equations are time reversible

$$(*)\begin{cases} \psi(t) & = & i\left(\psi_{ss} \pm \frac{1}{2}\left(|\psi|^2 - \frac{|a|^2}{t}\right)\psi\right) \\ \psi(1,s) & = & ae^{i\frac{s^2}{4t}} + \epsilon_1(s) \end{cases}$$

Q1 Solve $(*)$ for $0 < t < 1$ for reasonable (small) ϵ_1.

Q2 $\lim_{t\downarrow 0} \psi(t,s) = ?$

Conformal transformation

$$\psi(t,s) = \frac{e^{i\frac{s^2}{4t}}}{\sqrt{it}} v\left(\frac{1}{t}, \frac{s}{t}\right)$$

$$-v_t = i\left(v_{ss} \pm \frac{1}{2t}\left(|v|^2 - |a|^2\right)v\right)$$

Particular solution $\Phi_a = a e^{i\frac{s^2}{4}}$ $v_a = a$

$v_1 = a + \epsilon$

$\psi_1 = (a + \epsilon) e^{is^2/4}$

RENORMALIZED ENERGY

$$E(t) = \frac{1}{2}\int |v_s(t)|^2\, ds \mp \frac{1}{4t}\int \left(|v(t)|^2 - |a|^2\right)^2 ds$$

$$\frac{d}{dt}E(t) = \pm\frac{1}{4t^2}\int \left(|v|^2 - |a|^2\right)^2$$

LONG RANGE SCATTERING (V. Banica)

$$(**)\begin{cases} -v_t &= i\left(v_{xx} \pm \frac{1}{2t}\left(|v|^2 - |a|^2\right) v\right) \\ v(t,x) &= a + \epsilon(t,x) \end{cases}$$

Asymptotic profile (Ozawa, Hayashi–Naumkin)

$$v_1(t,x) = a + e^{\mp i \frac{a^2}{2} \lg t} e^{it\Delta} u_+(x)$$

for any u_+.

$$X_\gamma = \left\{ f : \; \|f\|_{L^2} + \left\| |\xi|^{2\gamma} \widehat{f}(\xi) \right\|_{L^\infty(|\xi|\leq 1)} \right\}$$

Theorem 1.– Given $0 < \gamma < 1/4$ and $a > 0$ there exists $\delta > 0$ such that if

$$\|v(\cdot, 1) - a\|_{X_\gamma} < \delta$$

then $\exists ! v$ solution of $(**)$ and $u_+ \in X_{\gamma^-}$ with

(i) $v - v_1 \in \mathcal{C}\big([1, \infty) : X_{\gamma^-}\big) \cap$ Strichartz

(ii) $\|v - v_1\|_{L^2} = \mathcal{O}\left(t^{-1/4}\right)$.

Hence

Q1 **YES**

Q2 **NO:**
$$\psi(t, s) = \frac{e^{is^2/4t}}{\sqrt{it}} v\left(\frac{1}{t}, \frac{s}{t}\right)$$

$$= \frac{a}{\sqrt{it}} e^{is^2/4t} + e^{-i\frac{a^2}{2}\lg t} \widehat{u}_+(s) + \mathrm{o}(t)$$

$$\psi_s \simeq \frac{s}{2t} \frac{a}{\sqrt{it}} e^{is^2/4t} + e^{ia^2 \lg t} \partial_s \widehat{u}_+(s)$$

$$\widehat{\epsilon}(t,0) = \int_{-\infty}^{\infty} \epsilon(t,x)dx$$

$$i\epsilon_t + \epsilon_{xx} = \pm\frac{1}{2t}\left(|\epsilon + a|^2 - a^2\right)(\epsilon + a)$$

$$\frac{d}{dt}\int |\epsilon + a|^2 - a^2 = 0 \qquad ; \qquad \int |\epsilon + a|^2 - a^2 = C_0$$

$$i\widehat{\epsilon}(t,0) = \pm aC_0 \lg t + \int_1^t \frac{1}{\tau} NL(\epsilon)d\tau$$

Hence $\left|\widehat{\epsilon}(t,0)\right| \geq \dfrac{a|C_0|}{2} \lg t.$!!

Theorem 2.– Let $X_1(s)$ be the curve obtained from

$$\psi(1,s) = (a+\epsilon_1)e^{is^2/4}.$$

Then, there exist a unique $X(t,s)$ solution of the B.F. for $0 < t < 1$ with $X(1,s) = X_1(s)$ and a unique $X_0(s)$ such that

$$\sup_s \left|X(t,s) - X_0(s)\right| \leq Ca\sqrt{t}.$$

Moreover there exist $T^{\pm}, E^{\pm}, T_0(s), T_0(0^+)$, and $T_0(0^-)$ such that

(i) $T(t,s) = T^{\pm} + \mathcal{O}\left(\frac{1}{s}\right), \quad s \to \pm\infty,$

(ii) $n - ib = E^{\pm} e^{is^2/4t + ia^2 \lg \frac{|s|}{\sqrt{t}}} + \mathcal{O}\left(\frac{1}{s}\right), \quad s \to \pm\infty,$

(iii) For $s \neq 0 \quad \lim_{t \downarrow 0} T(t,s) = T_0(s)$

(iv)

- $\lim_{s \to 0^{\pm}} T_0(s) = T_0(0^{\pm})$
- $T_0'(s) \in L^2$
- $\sin \frac{\theta}{2} = e^{-\pi \frac{a^2}{2}} + \mathcal{O}\big(\|v(\cdot,-1) - a\|\big)$

and θ the angle between $T_0(0^+)$ and $-T_0(0^-)$.

About the proof

$$T_s = \mathrm{Re}\,\overline{\psi} E$$

$$E_s = -\psi T$$

$$T_t = \mathrm{Im}\,\overline{\psi}_s E$$

- $$\begin{aligned} T^+ - T(t,x) &= \int_x^\infty T_s ds \\ &= \mathrm{Re} \int_x^\infty \frac{e^{-is^2/4t}}{\sqrt{t}} (a+\epsilon)\left(\frac{1}{t}, \frac{s}{t}\right) E(t,s) ds \end{aligned}$$

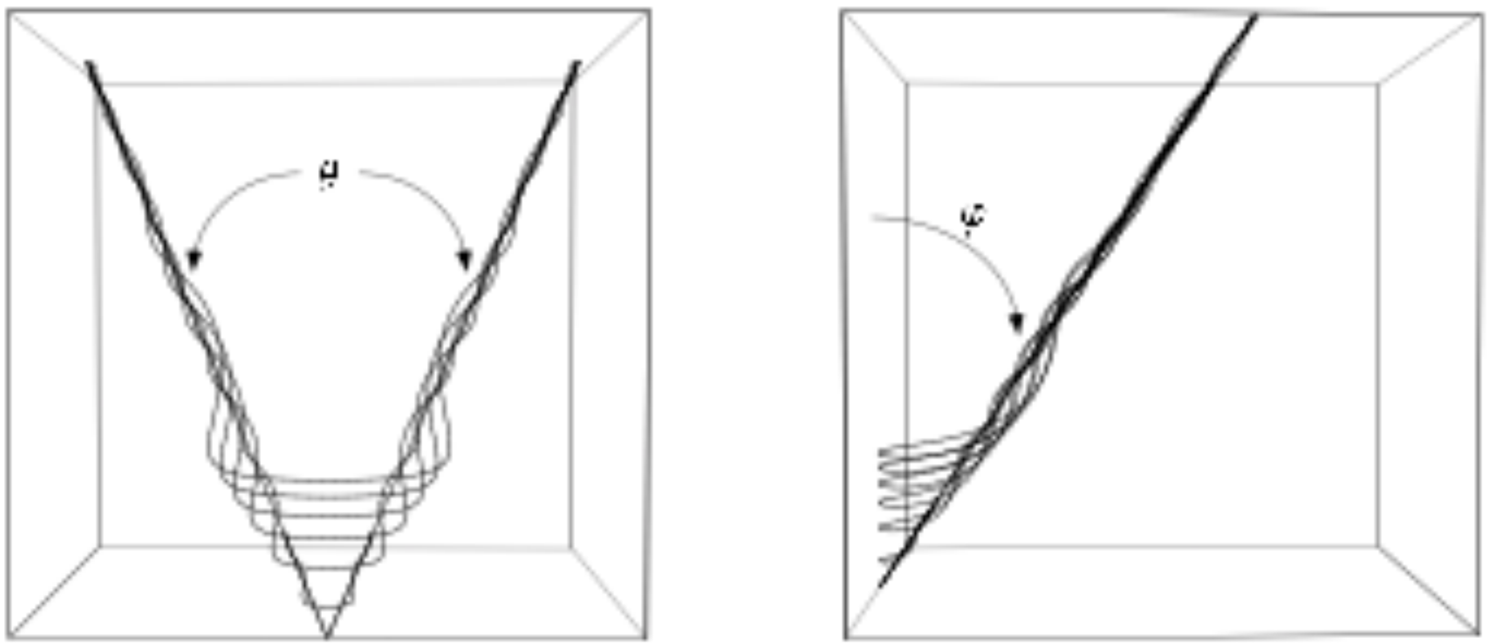
  + integration by parts
  + growth of $\|J\epsilon\|_{L^2}$ $\qquad J = x + 2it\partial_x$

- Fot T_t notice that the $\lg t$ correction of the phase of ψ_s and E cancel each other.

- $$T_0(x) = T^+ \quad + \quad \operatorname{Im} E^+ \int_x^\infty h(s)ds$$
$$+ \quad \operatorname{Re} \int_x^\infty h(s) \int_x^\infty \overline{h(s')} T_0(s')ds'ds.$$

$$h(s) = \widehat{u}_+ \left(\frac{s}{2}\right) \frac{i}{s^{ia^2}}.$$

Conclusions



(Banica, V. '07)

Theorem 3.– Assume in **Theorem 2** the extra hypothesis that $\widehat{u}_+ \in \dot{H}^{-2}$, then there exist $(T,n,b)(t,0)$ such that

$$\left|(T,\tilde{n},\tilde{b}) - (T_a,n_a,b_a)\right|(t,0) \leq \mathcal{O}\left(\|v_1 - a\|\right)$$

with

- $\tilde{n} + i\tilde{b} = e^{i\Phi/2}(n+ib)$
- $\dfrac{a^2}{t} + \Phi' = 2\left(\dfrac{c_{xx} - c\tau^2}{c}\right)(t,0) + c^2(t,0)$

Proof. Fefferman–Stein & Tsutsumi

$$\left\| \partial_x \left| e^{it\partial_x^2} u_0 \right|^2 \right\|_{L^2} = \left| \iint \left| \widehat{u}_0(\xi) \right|^2 \left| \widehat{u}_0(\eta) \right|^2 |\xi - \eta| d\xi d\eta \right|^{1/2}.$$

- Gustafson, Nakanishi, Tsai. (G-P.)

THANK YOU FOR YOUR ATTENTION