Contextual Semantics Machinery

Sahil Imtiyaz[†⊚] and Emanuela Merelli[‡]

[†‡]Department of Computer Science and Mathematics, University of Camerino, Italy
[⊚]Basque Center for Applied Mathematics, Spain

Abstract

The study establishes a formal connection between quantum contextuality and interactive computation.

Keywords — non-locality and contextuality, interactive computation, algebraic topology, simplicial complex.

1 Introduction

Richard Feynman's seminal lecture of 1981 at the *physics of computation* conference in Massachusetts Institute of Technology (MIT) centered around a key question of the possibility of simulating quantum physics, particularly a prominent non-classical feature of non-locality, by a universal Turing machine. The negative answer to this question became one of the starting points for the genesis of quantum computation and information science and a notable proposal for a quantum computer [16]. The field significantly focuses on speed-up solutions of a specific set of problems by offering non-classical features as resources. However, Feynman's reasoning was not limited to speed-ups but also on the notion of computation in the light of this modifying physical reality inspired by Edward Fredkin's discrete mechanics of information flow.

The community of theoretical computer science independently pursued this idea of physical genesis of computation which goes back to Petri whose net theory of concurrency was influenced by the idea of discrete spacetime [24]. One of the promising proposal in this direction is Milner's information dynamics – a structural theory of processes – to suggest study of fundamental structures of computation [21], further reinforced by Abramsky's vision of the calculus of a fundamental science of information dynamics [1, 4]. A relatable culmination of structure burgeoned from the interplay between geometry and concurrency theory [15]. The global structure possibly linking any far apart contexts could allow counter-intuitive *action at a distance* such as non-locality, for instance in bigraphs [26].

The idea of non-locality put forth its buds in interactive computation which could arise from the incompleteness of the observer to visualise all the contexts of the system simultaneously [28, 27]. Zuse suggested a qualitative structural explanation of such non-local correlations in cellular automata [29], unfolding from his work on calculating space, who along-with Petri and Feynman co-participated the 1981 conference at MIT. Alongside Yuri Manin's emphasis on the mathematical foundations of quantum computer in quantum foundation community, and the pioneering work of von Neumann's algebraic generalisation of quantum physics strongly suggest to access the mathematical foundations of quantum resources not only for speed-ups but also expressiveness of novel categories of computation.

One of the most promising steps in this direction is the work of Abramsky and Coecke on categorical axiomatics for quantum mechanics which provides a new mathematical foundation of quantum physics relevant to computer science [6] The framework could become a cornerstone to understand the fundamental structure of computation perhaps for a semantic revolution in the theoretical computer science [1, 4]. Of our particular interest is the phenomenon of quantum contextuality, a source of an extraordinary feature of entanglement, which serves as a theoretical bedrock for the second quantum revolution. The feature is proven real in the physical experiments and quantifiable as a resource in quantum algorithms to achieve quantum advantage.

The phenomenon of contextuality motivates the study of a mathematical structure independent of the formalism of quantum physics. In 2011, Abramsky and Branderburger proposed a new structure based on sheaf theory, which corresponds to topological obstructions to the existence of a global section responsible for local consistency and global inconsistency ($LC - GI$), known as *contextual semantics* [5]. This emerging semantics has initiated a step towards a general theory of quantum contextuality and provided new insights to the theory of relational database [3], natural language semantics [9], robust constraint satisfaction [7] and logic [2] in computer science. These latest results motivate us to search for its interpretation and applicability for interactive computation reflecting on a question: what could be an underlying model of computation that can express non-locality and contextuality?

The Bell-Kochen-Specker theorem already proved the impossibility of local hidden variables to explain non-locality and contextuality [12, 18], together with, invoking inclusion of global variables because *contextuality is topological* [14, 23]. As a result, the theorem paves a way to geometric models of computation unlike Turing machine whose tape structure recalls local hidden variables. The overall interaction among processes and resources in geometric models is represented as a simplicial complex, where every feasible local process always admits a global structure i.e., there is *always* local-global consistency, by avoiding (non-trivial) loops in the simplicial complex quite central to concurrency theory [15]. The loops are avoided due to causality conditions of the structure.

We discover that this cyclicity (non-trivial loops) quantifies contextual semantics ($LC-GI$) which could be allowed without violating causality, emerging through a synergistic interaction between computation and global cumulative structure of contexts, woven together as as topological structure of the environment. The structure constrains the computation which in turn could change its topology – the concept of openness. A loop in the structure depends on the corresponding path in its associated computation space; which makes loops causally explainable in this dynamic interaction. The concept of openness captures the fundamental concept behind non-locality and contextuality, that is, not all computations are feasible in a context-sensitive non-linear environment; which could serve as lineament towards expressiveness of novel categories of computation. The computation-structure interaction is described by a model of computation, topological interactive machine (TIM), whose structural isomorphism to fiber bundles have already been explored in gauge theory in the light of non-locality [22]. The theory behind the computational and mathematical modelisation of our work is the topological field theory of data proposed by Rasetti and Merelli which has proved effective to extract hidden emerging patterns, like empirical feature of quantum contextuality, existing among data which is part of the global data space, alongside offering a potential semantics based on physical field theories to express these patterns [25, 20]. The model can express all the levels of different degrees of correlation ranging from non-locality in Hardy model to strong contextuality in Kochen-Specker model. We restrict to the expressiveness of contextual semantics machinery which could invoke a deeper investigation into the foundations of computability based on interaction as observation.

Our Contributions The paper introduces a formal model of computation, topological interactive machine, whose mathematical structure is isomorphic to the fiber bundle to encode contextual semantics through its structure-computation interaction. The formalisation is an application of sheaf-theoretic modelling of non-locality and contextuality for interactive computation. The correspondence facilitates the concept of openness in computation unlike the Turing machine whose tape based structure recalls local hidden variables that cannot reproduce the non-locality and contextuality due to Bell-Kochen-Specker theorem. We prove that TIM shows behavioural bisimilarity to empirical models of quantum physics. The mathematical semantics expressing contextual semantics machinery in TIM is the gauge group which is a semi-direct product of symmetry group and fundamental group representing computation and structure respectively. The symmetry group associated with a polyhedron representing computation is subjected to structural constraint of its associated discrete space. The structure constrains the (in)admissibility of symmetry transformations of polyhedron. A local section would correspond to any possible permutation of the polyhedra and the global section corresponds to permutations that permute a context back to itself in the presence of structural constraint. For instance, for Kochen-Specker model no such family of global sections exist because the model is strongly contextual. The topological structure of the environment encoding contextual semantics is further explored as *strong collapse* of its combinatorial structure using discrete Morse theory (DMT). The work constructs discrete topological spaces associated with the empirical models. This bundle description to characterise contextuality in the light of obstruction theory was put forth as an impelling question by Carù to seek cohomological obstructions as a problem of constructing cross-section of a bundle [14]. We further apply the results based on bundle description on the examples of empirical models in the foundations of quantum physics. The examples include Hardy model, Peres-Mermin square, Kochen-Specker model, Popescu-Rohrlich Boxes and Greenberger-Horne-Zeilinger Model. Contextual semantics quantifiable as $LC-GI$ emerges in the discrete structure of TIM as transitory virtual loops that are indicators of changing homotopy class of the space facilitating openness and in this sense could be characterised as a symmetry breaking process.

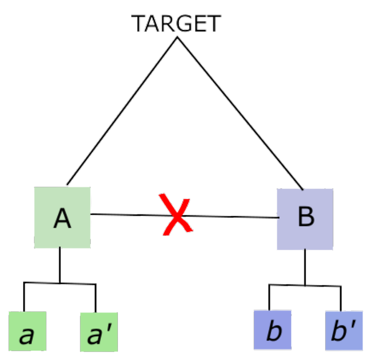
Organisation We introduce the setting for understanding the background of contextual semantics and topological model of computation along with their interplay in Section 2. Afterwards, we introduce TIM as a model of computation which is isomorphic to the fiber bundle structure in Section 3, along with its behavioural bisimilarity to empirical models of quantum physics in Section 4. We give main results of the study in Section 4 and Section 5. Throughout Section 6 we test the results on the examples of empirical models in TIM framework based on results of Section 4 and Section 5. Section 7 provides discussion on promising future work along with helpful supplementary material in Appendix A.

2 The Setting

This section introduces a basic idea of contextual semantics and its sheaf-based modelling. The computing synthesis of non-locality and contextuality in sheaf-based approach turns out to be synergistic computation-structure interaction, i.e., the (topological) structure constrains the computation which in turn could change its topology – openness. As a result, we introduce an example of topological model of computation to solely elucidate the requirement of additional structures and concepts to express the contextual semantics machinery.

2.1 Contextual Semantics

Given two agents A and B, which are space-like separated over a distributed network; both agents have two local bit registers: A with a and $a^{'}$; B with b and $b^{'}$; each of them can store values either 0 or 1. When the registers receive a value (0 or 1) from some (hidden) source in the background, the agents observe and transmit it to some target. For example, A can *choose* its register $a^{'}$ and *observe* 0 and simultaneously B can *choose* b and *observe* 1, which comprises basic event of the system. The frequency of similar events represent the probability of each event as shown in the table on right side along with its basic set-up on left side of Figure 1. Note that each probability p_i is greater than zero.



A,B	(00)	(01)	(10)	(11)
(a,b)	p_1	p_2	p_3	p_4
(a,b')	0	p_5	p_6	p_7
(a',b)	0	p_8	p_9	p_{10}
(a',b')	p_{11}	p_{12}	p_{13}	0

Figure 1: On the left is the basic set-up and on the right is the probability distribution table of all events.

The entry in the table means, if A looks at a and B looks at $b^{'}$, then the probability that A sees 0 and B sees 1 is p_5. $\{a, a^{'}, b, b^{'}\}$ is the measurement set, $\{(a, b), (a, b^{'}), (a^{'}, b), (a^{'}, b^{'})\}$ are the measurement contexts, in short contexts, and 0 and 1 are the outputs observed.

Instead of probabilities, one can talk about possibilities of events, i.e. if the table has some probability p_i (greater than zero) between any contexts, then we assign 1 to the corresponding entries of the table, otherwise 0, as shown in table (here, of the Hardy model) on the left of Figure 2. The table has a geometric realisation; with (square) base representing the contexts associated with fibers which represent outcomes; the bundle diagram, as shown in the right side of Figure 2. Its corresponding planar (linearised) diagram is shown in Figure 3 for a better view. Note that in general the square base representing contexts can be realised as a simplicial complex $\mathcal{K}$.

The base of the bundle diagram is a rectangle, $\mathcal{K}$ in more general scenarios; consisting of the measurement set $\{a, a^{'}, b, b^{'}\}$ and the fibers consist of possible outputs i.e. 0 and 1. To each context of the measurement set, one assigns its two output values (0 and 1) as a fiber associated with it. Based on any context from the table, there is an edge between their corresponding outputs in the fibers, if and only if, it is a possibilistic event with value 1 in the table. For example, if one chooses context $(a^{'}, b^{'})$ at (11), i.e, output of $a^{'}$ is 1 and that of $b^{'}$ is 1 from table in Figure 2, then there is no edge between their corresponding fibers because the event is a non-possibilistic event as

A,B	(00)	(01)	(10)	(11)
(a,b)	1	1	1	1
(a,b')	0	1	1	1
(a',b)	0	1	1	1
(a',b')	1	1	1	0

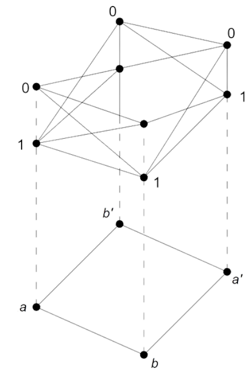


Figure 2: On the left is possibilistic table and on the right is its corresponding bundle diagram: base consisting of contexts and fibers consists of outputs.

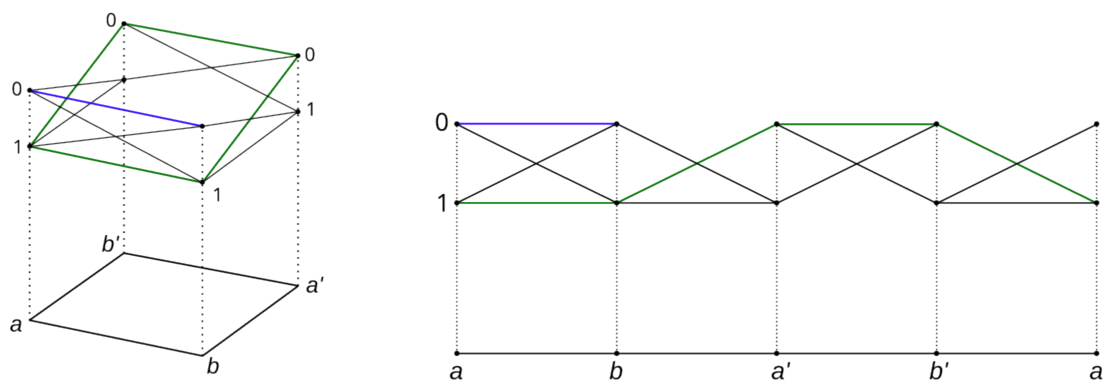


Figure 3: Planar diagram of Hardy Model.

shown at corresponding entry of the table. One closes edges between values in fiber if they can appear together as possible joint outcome over $\mathcal{K}$. For example, the green rectangle in the fiber space of Figure 3 corresponds to four possibilistic contexts; (a,b), (a',b), (a',b') and (a,b') of the table at outputs $(1,1)$, $(0,1)$, $(0,0)$ and $(1,0)$ respectively. It means, all the contexts are simultaneously satisfiable at these outputs which corresponds to a *closed loop* in the fiber space.

Remark. *It is worth noting that the table is not randomly generated. It is based on various physical and mathematical constraints. For example, the sum of probabilities for each context subject to different outputs should be equal to one;* $p_1+ p_2+ p_3+ p_4 = 1$. *So, the table cannot reproduce* any *behaviour, e.g., in which all entries are zero. Moreover, if we change one of the entries of the table it affects the table as a whole. The physical constraint in the background includes relativistic constraint and choice of agents to select the measurement setting independent of background source (hidden variables), which correspond to no-signalling and lambda independence respectively in sheaf-based approach. The table is the synthesis of background observational data from a physical Bell-type experiment.*

There is one context (a,b) in the table which is locally possible at (00) but globally not possible, i.e. when we extend it through different contexts over the fiber space denoted in blue in Figure 3, it doesn't form a closed loop because the contexts over its base space $\mathcal{K}$ are not simultaneously satisfiable which is expressed as non-collapsibility of $\mathcal{K}$. The table expresses three class of behaviour; possibilistic behaviour, non-possibilistic behaviours and the $LC-GI$ behaviour.

Local consistency means if an outcome from A point of view is possible in a given context then it should stay possible even if B were to change which measurement it makes, i.e., free choice of measurement. It means there is always a way to continue from any edge over the base of bundle diagram. For example, if A chooses a then whatever B chooses, whether b or $b^{'}$, there is an edge between both possibilities. Global consistency means one can form a complete closed path in fiber space that assigns a unique value to each context, which means all selected contexts are simultaneously satisfiable. For example, the green line in Figure 3 forms a loop.

Remark. *The loop in the fiber space is possible when its corresponding base space is simply-connected, i.e., the simplicial complex representing the contexts is acyclic implying collapsibility; meaning all the contexts are simultaneously satisfiable, which infer locality/non-contextuality; otherwise, non-collapsible property of simplicial complex imply holes expressed in non-trivial (co)homological group due to cohomological obstructions quantifying contextuality and non-locality. This consistent families of contexts and their extension emerges from Vorob'ev theorem.*

One of the relevant questions to ask is: what could be the possible hidden source in the background that can explain/reproduce the behaviour of these models? It was a central question of Einstein-Bohr debate on the existence of hidden variables. Bell theorem concludes that no classical source/local hidden variables can reproduce such behaviour. Classical Turing-like interactive models cannot express this category of behaviour/computation due to Bell-Kochen-Specker theorem; which recalls the linear tape structure of their environment as local hidden variables. Then, *What could be the underlying model of computation that can express the behaviour of the Bell-like models of quantum physics?* Precisely, *what could be computational generalisation of these empirical models based on observational data?* A possible explanation for $LC-GI$ is a topological representation of the linear environment of Turing machine to include global context dependency during computation. The linear base space of bundle diagram is replaced by a topological representation of environment as shown in Figure 4 which is formalised throughout the paper as TIM computational framework.

A,B	(00)	(01)	(10)	(11)
(a,b)	1	1	1	1
(a,b')	0	1	1	1
(a',b)	0	1	1	1
(a',b')	1	1	1	0

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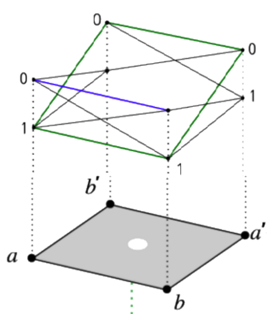

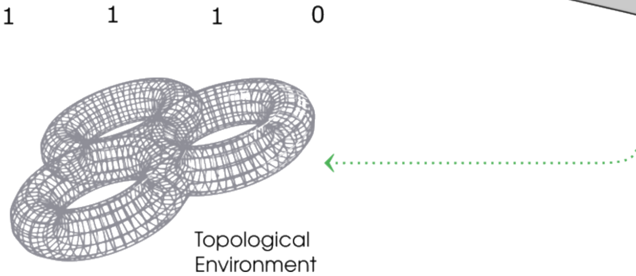

Figure 4: Contextuality is a topological property which advocates a topological environment in the background as a source to explain this novel category of computation, i.e., $LC-GI$.

2.2 The Sheaf Modelling

The observational data from the above Bell-like experiment fits naturally in sheaf-based modelisation. We give a rigour free background of sheaves and its generalisation as well as interplay with the empirical model. The interested readers can refer to references [5, 8] for further extensive study of the subject. This subsection formalises the above ideas of the bundle diagram of Figure 2 in the sheaf-theoretic machinery.

Basic idea of sheaf theory Sheaves comprise of two components that operate together: the horizontal topological component over which its vertical algebraic-type resides. A basic example is sheaf *of* vector space *on* abstract simplicial complex. The individual sheaf is called a stalk. The idea is to move between vectors in each vectors space (stalk) via linear maps in the vertical component together moving from higher simplices to 0-simplex on its horizontal topological component. The base space in Figure 2 is generalised to a combinatorial structure realised as simplicial complex and the fibers are generalised to an algebraic structure, say, a ring or a field.

Suppose X is a topological space with open set $\mathcal{U}$ *covering* X. Let $\mathcal{U}_i$ be open subsets of $\mathcal{U}$, where i denotes the number of open subsets of $\mathcal{U}$ in order to cover it. Sheaf theory operates on a function r defined on $\mathcal{U}$ of X and restricts to functions $r|_{\mathcal{U}_i}$ defined on $\mathcal{U}_i \subset \mathcal{U}$. $\mathcal{U}$ is recovered by gluing together the restrictions to $\mathcal{U}_i$. The *restriction-gluing* is main theme of sheaf theory. A (pre)sheaf $\mathcal{O}$ on X can be described as a rule/function which assigns to each point, a set O consisting of the elements of the functions at that point defined by its neighbourhood, i.e., $\mathcal{O}(\mathcal{U}) := O^{\mathcal{U}}$. The sets $O^{\mathcal{U}_i}$ for all $\mathcal{U}_i$ can then be glued together by a suitable topology so as to form bundle projected onto X. The individual *nice* function for this sheaf is then a cross-section of the projection of this bundle. The combinatorial representation of X is a finite simplicial complex $\mathcal{K}$. When X is discrete, the presheaf $\mathcal{O}$ is trivially a sheaf. In fact, every presheaf can be uniquely realised as a sheaf. A section of a sheaf is an element $\kappa \in \prod_{\sigma,\tau \in \mathcal{K}} \mathcal{O}(\sigma)$ such that $\mathcal{O}(\sigma \to \tau)\kappa(\sigma) = \kappa(\tau)\ \forall\ \sigma \subseteq \tau$, i.e., σ is the face of τ and $\kappa(\sigma)$, $\kappa(\tau)$ are elements of O. The map $\mathcal{O}(\sigma \to \tau)$ is a linear map that transforms matrix $\kappa(\sigma)$ to $\kappa(\tau)$ over vertical component of sheaf given its corresponding collapsible or acyclic $\mathcal{K}$, i.e., the cochain complex is exact. The *cohomological obstructions* which arise in sheaf theory is the main mathematical concept to express phenomena of non-locality and contextuality. The obstructions arise due to non-simply connected regions of the topological space which has non-zero or non-vanishing cohomology groups. There could locally exist a section κ via linear map but globally non-extendible due to non-simply connected X.

Generalisation of empirical models through sheaf structure The sheaf theory naturally structures these Bell-like models. The above mathematical account of sheaves naturally fits the bundle diagram of Figure 2. Here, we give a brief interplay between structural and conceptual aspects of sheaves and the empirical model.

Quantum contextuality restricts the simultaneous assignment of values to all measurements. This feature is captured by introducing the measurement cover $\mathcal{U}$ which covers the space X. The subsets of $\mathcal{U}$ denoted by $\mathcal{U}_i$ represents the set of compatible families of measurement contexts, i.e., those which can be measured simultaneously with, $\bigcup_{\mathcal{U}_i \in \mathcal{U}} \mathcal{U}_i = X$. The combinatorial description of X is represented to be a simplicial complex $\mathcal{K}$ which is homeomorphic to X. It comprises of finite measurement set representing all the available measurements in a general experiment. Each measurement $\mathcal{U} \in X$ produces an outcome in a set $O_{\mathcal{U}}$. The set O turns out to be output set in sheaf generalisation of empirical model. The measurement set, the measurement contexts and the outcome set constitute the measurement scenario. X is equipped with a discrete topology and defines sheaf of events which relates open subsets $\mathcal{U}$ of X to the space of sets O.

A *section over* $\mathcal{U}$ is a function $\kappa : \mathcal{U} \to O$ that describes the event in which the measurements in $\mathcal{U}$ were performed, and the outcome $\kappa(m)$ was *observed* for each $m \in \mathcal{U}$. The function κ is generalised mathematically in a *sheaf structure* that assigns set of sections over O to each *measurement context* $\mathcal{U}$. It is represented topologically as a bundle diagram as shown in Figure 3. There exists a unique section for a family of subset in $\mathcal{U}$ if its corresponding family of sections is *compatible* and has simultaneous existence i.e., $\kappa_i|_{\mathcal{U}_i \cap \mathcal{U}_j} = \kappa_j|_{\mathcal{U}_i \cap \mathcal{U}_j}$. For brevity, the non-collapsibility of the simplicial complex constrains the extendability of contexts to global section due to cohomological obstructions [8]

Measurement scenarios and the sheaf of events define the experiment setting and are therefore independent of any physical system. The application of this sheaf scenario on an actual physical system is captured by the notion of empirical model. The empirical model **e** in this sense is a compatible family $\{\mathrm{p}_{\mathcal{U}_i}\}_{\mathcal{U}_i \in \mathcal{U}}$, where $\mathrm{p}_{\mathcal{U}_i}$ is a probability distribution on $\mathcal{O}(\mathcal{U})$. The compatibility of the family is independent of the choice of $\mathcal{U}$; **e** is possibilistic expandable if and only if every section is a member of compatible family. It is possibilistically non-extendable if for some section κ (cross-section), there is no such family. **e** is strongly contextual if for every κ there is no such family. A simple understanding of this mathematical setting is comprehensively expressed in the tabular representation of the empirical models, for instance, the Hardy table shown in Figure 2.

2.3 Computing Synthesis of Contextual Semantics

Notice, in sheaf modelling there is a interplay between horizontal and vertical component; a section in fiber space depends on the contexts in the topological space. *The computing synthesis of contextual semantics is the synergistic* *interaction between structure and computation*, i.e, the topological structure constrains the computation which in turn could change its topology – *openness*.

The global context dependency in computation is a subject matter of interplay between algebraic topology and concurrency community. The overall execution of the interaction among processes is represented as simplicial complex in a topology-based model of computation [15]. Here, our *sole* aim is to elucidate the additional structure and concepts that are required in these topological models to express contextual semantics with the help of an example without addressing the community in general. Our focus is on the class (type) of computation/behaviour expressible using these topology-based models.

Consider a process specification $(\mathrm{P_a};\, \mathrm{P_b};\, \mathrm{V_b};\, \mathrm{V_a}) \,||\, (\mathrm{P_b};\, \mathrm{P_a};\, \mathrm{V_a};\, \mathrm{V_b})$, where $\mathrm{P_a}$, $\mathrm{P_b}$ or $\mathrm{V_a}$, $\mathrm{V_b}$ means locking or releasing a resource a and b respectively. The topological space associated with the overall process execution, known as *swiss flag*, is shown in Figure 5.

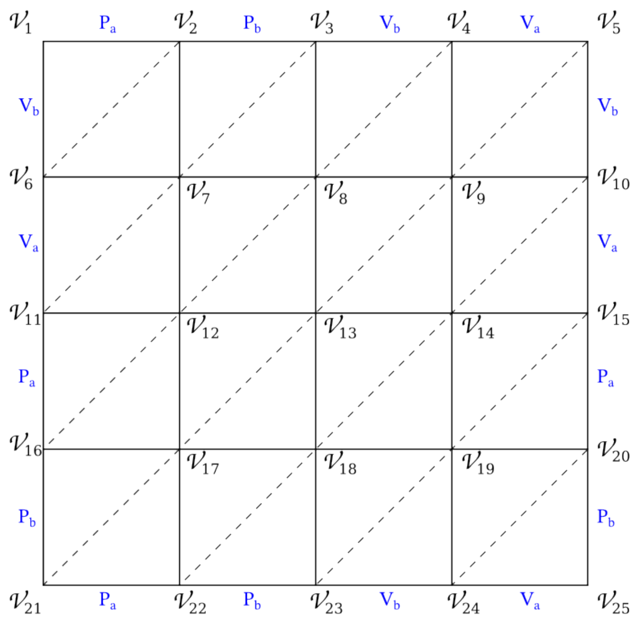

Figure 5: Topological space associated with the process execution.

The diagram in Figure 5 is the discrete space of all possible executions of the given processes. It consists of the computation $\mathcal{V}_1 \cdot\, \mathrm{P_a} \rightarrow \mathcal{V}_2$, the commuting contexts executing concurrently like $\{\mathcal{V}_1\mathcal{V}_2\mathcal{V}_6\mathcal{V}_7\}$ and the global contexts that forms the overall structure *woven* through all contexts. One could defines a resource potential res to have maximum capacity of say 1; so, $res(\mathcal{V}_1)a = res(\mathcal{V}_1)b = 1$ in the beginning. Since, the transition $\mathcal{V}_1 \rightarrow \mathcal{V}_2$ is labelled P_a, and the action of P_a is to decrement the number of resources of a, so $res(\mathcal{V}_2)a = 0$ and $res(\mathcal{V}_2)b = 1$. Following the parallel execution in this way allows four squares $\{\mathcal{V}_1\mathcal{V}_2\mathcal{V}_6\mathcal{V}_7\}$, $\{\mathcal{V}_4\mathcal{V}_5\mathcal{V}_{10}\mathcal{V}_9\}$, $\{\mathcal{V}_{21}\mathcal{V}_{22}\mathcal{V}_{17}\mathcal{V}_{16}\}$ and $\{\mathcal{V}_{19}\mathcal{V}_{20}\mathcal{V}_{25}\mathcal{V}_{24}\}$ to have $res = 1$ given the order of executions. All executions are infeasible due to unavailability of resources except the above four contexts. The infeasible and feasible computation correspond to possibilistic and non-possibilistic behaviour of empirical models.

The model cannot explain this new category of computation (LC-GI) because every feasible local process *always* admits a global structure, i.e., there is always local-global consistency, by avoiding (non-trivial) loops in $\mathcal{K}$ quite central to concurrency theory. The loops are avoided due to causality conditions on the structure.

We discover that this cyclicity (non-trivial loops) is responsible for the new category of computation (LC-GI) which could be allowed without violating causality, emerging through a synergistic interaction between computation and global cumulative structure of contexts, woven together as as topological structure of the environment. The structure constrains the computation which in turn could change its topology – openness; a loop in the structure $\mathcal{K}$ depends on the corresponding path in its associated computation space; which makes loops causally explainable in dynamic structure. This computation-structure acts as a unique algebraic object expressing this novel category of computation. The idea formalises into topological interactive machine formalised in Section 2;

which is a transition system interacting/computing in a structural open environment; in which process description of environment is switched to a structural (topological) representation.

There are two key observations about the above discrete topological space of Figure 5 that are worth nothing here:

1. Firstly, the discrete space is not constructed from the observational data output from a Bell-like physical experiment; where there is an effective interaction between the observer and the microcosm through photon as agency of observation. The observational data is not known a priori to the observer, but rather discovered by observation which is essence of Bohr complementarity and Heisenberg's principle of uncertainty; quantifiable in data as empirical phenomenon of quantum contextuality. The space in Figure 5 is based on a priori defined program not data unlike empirical model of quantum physics — *non-empirical.*

2. Secondly, the discrete space associated with the program expresses all of its possible executions. The possibility to know well defined a priori executions of a process is because the structure of space doesn't change — *static.*

We address the first point by constructing a discrete topological space from PTM computation using topological data analysis from the information provided in tables of the empirical models. It is not based on a priori execution of program but data. The observational data output from the Bell-like experiments is generalised and condensed in tables of empirical models. The construction of discrete space in our work is based on the observational (empirical) data from the tables.

The second point is addressed by allowing computation to change the topology of the constructed discrete space which in turn constrains the computation in a feedback loop. The interplay between global topology of the contexts and the computation allows non-trivial loops to *emerge* in a causal way and facilitates the concept of an *openness* responsible for $LC - GI$. These loops are transitory responsible for changing the homotopy type of the space – a symmetry breaking process as discussed throughout particularly in subsection 5.1.

2.4 The Roadmap

The first step is to formalise an interactive machine that can express contextual behaviour in Section 2. The second step is to show that this machine can show behavioural bisimulation to empirical model in Section 3. The third step is the expressiveness of contextual semantics machinery in Section 3. The fourth step is to emulate examples of empirical models of quantum physics over the machine in Section 4. Models including, Hardy model in Subsection 6.2, Kochen Specker model in Subsection 6.3, Mermin-Peres Magic Square in Subsection 6.6, Popescu-Rohrlich Boxes in Subsection 6.5 and Greenberger-Horne-Zeilinger model in Subsection 6.4.

3 The Machinery

This section introduces the model of computation for expressing contextual semantics machinery. We start with a basic example to understand the working of the machine along with its formalisation and isomorphism with mathematical structure of the fiber bundle.

3.1 Topological Interactive Machine

Persistent Turing machine (PTM) is a minimal extension of Turing machine which captures the idea of interactive computation. It is a multi-tape Turing machine with a persistent work tape whose content is preserved between successive Turing machine computations. The *work tape* is a non-observable environment but affects the PTM sequential computation. PTM's environment is a mapping from PTM computation to its corresponding feasible set of equivalence classes which does not take into account the global contexts during its computation. As a result, PTM cannot express contextual behaviour even though conceptual non-observability of environment provides a qualitative description of contextuality.

First, the work-tape acting as an environment for the PTM computation could be given a topological meaning using topological data analysis in order to quantify contextual semantics. For brevity, consider each function q has a probabilistic measure to transform input i to output o based on the context. The PTM computation consisting of $q : i \to o$ produces empirical data in a given time t interval. Each $q_t : i \to o$ outputs *data* that is represented as an element in an arbitrary set Z. This *set* Z can be analysed by persistent homology, a procedure used in topological data analysis, to construct a topological (space) environment $\mathcal{E}$[13]. The *data* in this paper is observational data from a physical experiment of empirical models of quantum physics compactified in a tabular form...

Second, this topological space would further require its interplay with the computation as a unique algebraic structure to express $LC-GI$ as explained in the Section 2.3. In order to allow computation to change the topology of contexts and vice-versa, we need to explicitly *extract out* the computation from the structure. The abstraction is carried out by associating a *state space* $\mathbf{S}$ (consisting of states) to each node of the discrete topological space, here simplicial complex $\mathcal{K}$. It gives an element of (free) choice to each node to make any possible transition between state spaces. A transition between states in the state spaces is only permissible when their underlying simplices allow it. The computable function is generalised to partial function which *discovers* the transition in the presence of contexts. The computation is further allowed to affect the simplices of the simplicial complex which in turn could changes its topology. The partial function that transforms input i to output o at given time t is characterised by its corresponding set of path over $\mathcal{E}$.

The topological environment realised as simplicial complex encodes global contexts that represent structure constructed from PTM computation which represent function. Both function and structure are described as a unique algebraic object which has a mathematical description of a fiber bundle.

Remark. *The PTM is isomorphic to a general class of effective transition systems called interactive transition systems. Here, the process representation of the environment is replaced with structural/topological representation. TIM is a transition system interacting and computing in a structural open environment.*

Definition 1. *(**Topological Environment** [20]) The topological environment $\mathcal{E}$ is the simplicial complex $\mathcal{K}$ which is constructed from the set Z consisting of the PTM computations between states in the state space $\mathbf{S}_t$ available at a given time t using topological data analysis.*

First, we would like readers to refer to Appendix A.1 to understand the structure of fiber bundles with a basic example. We have dissected the different mappings of bundle for clarity. Here, we first give an example of TIM to clarify its working and after provide a formal definition.

3.2 Illustration of Working of the Topological Interactive Machine

We give a simple example to understand the working of the machine as well as meaning of symbols used in the definition 2 of the machine.

Given a simplicial complex $\mathcal{K}$ with four 0-simplices $\{\mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_3, \mathcal{V}_4\}$, five 1-simplices $\{\mathcal{V}_{12}, \mathcal{V}_{14}, \mathcal{V}_{23}, \mathcal{V}_{13}, \mathcal{V}_{34}\}$ and two 2-simplices $\{\mathcal{V}_{123}, \mathcal{V}_{134}\}$, as shown in the Figure 6. Here, $\mathcal{V}_{12}$ means an edge between $\mathcal{V}_1$ and $\mathcal{V}_2$ and similarly for other notations. Notice, there is a colour coding of each simplices $\mathcal{K}_i \in \mathcal{K}$: Green and Red. Here, we do not bother about *how* to colour code them so as to abstract away the complexity. The colouring is based on the information contained in the empirical models which are extensively discussed in Section 6. Instead, here we assign weight to each simplices $\mathcal{K}_i$.

$$w(\mathcal{K}_i) = \begin{cases} 1, & \text{if } \mathcal{K}_i \text{ is generic simplices: Green} \\ 0, & \text{if } \mathcal{K}_i \text{ is critical: Red} \end{cases}$$

Each vertex of the $\mathcal{K}$ has an additional information which is represented as a fiber $\mathbf{S}_i$ attached to each vertex $\mathcal{V}_i$ via projection map π_i, i.e., $\pi_i : \mathcal{V}_i \rightarrow \mathbf{S}_i$. In the given example 6, we have four fibers (state spaces) $\{\mathbf{S}_1, \mathbf{S}_2, \mathbf{S}_3, \mathbf{S}_4\}$, with $|\mathbf{S}_1| = 2$, i.e., $\mathbf{S}_1$ consists of two states s_1 and s_2. Similarly, $|\mathbf{S}_2| = 3$,$|\mathbf{S}_3| = 1$ and $|\mathbf{S}_4| = 1$. We also set minimum bound to each $\mathbf{S}$ to be able to compute, here $|\mathbf{S}| > 1$, i.e., every $\mathbf{S}_i$ should contain at least two states to take part in computation. For example, $|\mathbf{S}_3| = |\mathbf{S}_4| = 1$; meaning they cannot take part in computation, so their corresponding simplices $\mathcal{V}_3$, $\mathcal{V}_4$ and $\mathcal{V}_{34}$ are set red colour. In our example, we have two checks; first is to observe $w(\mathcal{K}_i)$ for action A between any two state spaces and second is the cardinality bound of $|\mathbf{S}| > \mathbb{N}$ affecting $w(\mathcal{K}_i)$. It means, computation between two state spaces is constrained by $\mathcal{K}$ and vice-versa, i.e., cardinality less than minimum set bound of the state space could set $w(\mathcal{K})$ to zero.

We can write π instead of π_i and distinguish it with the associated set of $(\mathcal{V}_i, \mathbf{S}_i)$, i.e., one can say the map π at the particular $\mathcal{V}_i$ and $\mathbf{S}_i$; but here in the given example, we prefer subscript with π to make it more clear. In the formal definition of the machine we skip using the subscript, such as π_i meaning a map from S_i to V_i, because the subscript is implicit as one could talks about a particular fiber at a specific vertex.

The inverse map of π of any state $s_i \in \mathbf{S}_i$ is a particular vertex $\mathcal{V}_i \in \mathcal{V}$, i,e., $\pi^{-1}(s_i) = \mathcal{V}_i$. The embedding *Emb* of any inverse projection of π corresponding to a particular $s \in \mathbf{S}$, i.e. $Emb\{\pi^{-1}(s_i)\}$ is the set of all $\mathcal{K}_i$ of $\mathcal{K}$ having projection of $\pi^{-1}(s)$ as its element.

Here, lets choose $\mathbf{S}_1$, first we would check $|\mathbf{S}_1| = 2$, which satisfies our given bound on $\mathbf{S}$ ($|S| > 1$). Moreover, $\pi^{-1}(s_1) = \mathcal{V}_1$, which means $Emb(\mathcal{V}_1) = \{\mathcal{V}_1, \mathcal{V}_{12}, \mathcal{V}_{13}, \mathcal{V}_{14}\}$. Now, check the $w(\mathcal{V}_i)$, here, weight of all simplices is 1, i.e., $Emb(\mathcal{V}_i) \cap Red = \emptyset$. It means, state s_1 can make transition to any of the states of $\mathbf{S}_2$, i.e., there is an admissible action, say $a_{15} \in A$ between $s_1 \xrightarrow{a_{15}} s_5$.

- $\mathcal{V}$ is a set of vertices of a simplicial complex $\mathcal{K}$, and each $\mathbf{S}_i$ is a finite set which is a fiber associated with each $\mathcal{V}_i \in \mathcal{V}$ via projection map π; but π is not a bundle map from union of all $\mathbf{S}_i$ to the set of vertices $\mathcal{V}$. π further identifies the set of simplices (in its open neighbourhood) associated with its particular fiber/vertex via embedding function to decide whether the set contains any red simplex; and prepare for next step which is action between fibers, i.e., if there exists a red simplices in embedding of π, then the corresponding action would be infeasible, otherwise feasible. It could be equivalent to a bundle map in trivial fiber bundles; where the space is simply connected, i.e., its associated simplicial complex is collapsible. The bundle map in fiber bundle is of the type (π, ψ) or (π, σ), as shown in the Figure 30; with π and ψ having similar meaning as π and A.
- The action A can be seen in two equivalent ways; as a set of relations or set of pair of states. Since A depends on $w(\mathcal{K}_i)$ and $|\mathbf{S}|$, so we prefer to focus on the pair of states.

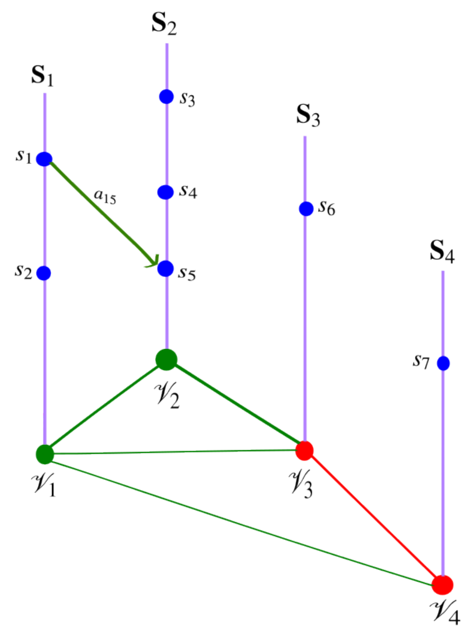

Figure 6: An Example.

After action A, here a_{15} between s_1 and s_5, both conditions are re-checked; i.e., $|\mathbf{S}|$ and $w(\mathcal{K}_i)$. Here, $|\mathbf{S}_1| = 1$, $|\mathbf{S}_2| = 2$ and hence, $w(\mathcal{V}_1)$ turns red which means in next iteration it cannot take part in the computation. This is the role of δ map.

- The map δ is a map from $\mathcal{V} \times \mathbf{S} \to \mathcal{V} \times \mathbf{S}$; which takes vertex in $\mathcal{K}$ and its respective fiber $\mathbf{S}$ to another vertex and fiber; $\delta_1 : \mathcal{V}_1 \times \mathbf{S}_1 \to \mathcal{V}_1^{'} \times \mathbf{S}_1^{'}$ and $\delta_2 : \mathcal{V}_2 \times \mathbf{S}_2 \to \mathcal{V}_2^{'} \times \mathbf{S}_2^{'}$ as shown as general case in Figure 7. δ map updates each fiber and its corresponding simplex based on global consistency condition; which is shown as purple dotted lines in Figure 30 between two different local diffeomorphisms in the fiber bundle. The updating process of δ and its comparison to fiber bundle can further be referred to figure 12, such that the squares in figure 8 commutes. It is like a simplicial map preserving fibers, taking a region in the total space and glues it together globally.

The square in left hand side of the figure 8 is the pre-conditions, the *local consistency* as in the example 6; which is to use projection maps and observing action. Fix a point in the square, say $\mathcal{V}_j$, one can project the image of $\mathbf{S}_i$ via π_i reaching $\mathcal{V}_i$ and then reaching $\mathcal{V}_j$ via $\mathcal{V}_{ij}$ (edge is green); or $\mathbf{S}_i$ makes an action A_{ij} which outputs

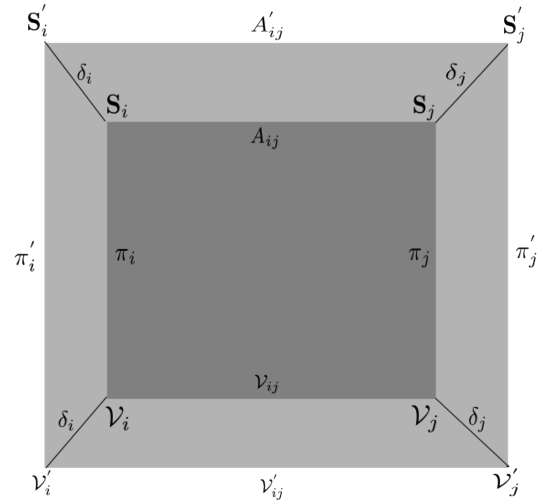

Figure 7: A computation step of TIM which can be iterated into possibly infinite computational steps as shown in Figure 9.

$\mathbf{S}_j$, which can further project its image via π_j finally reaching $\mathcal{V}_j$, i.e., $\mathcal{V}_{ij} \cdot \pi_i = \pi_j \cdot A_{ij}$, once conditions are satisfied. The square on right commutes of same Figure 8 is updating step, checking the post-condition, the *global consistency*. With slight abuse in notation, (π_i, A_i) sends $\mathcal{V}_i \times \mathbf{S}_i$ to $\mathbf{S}_i$ via A_i, as well as projects the image $\mathcal{V}_i$ via π_i. Similarly, $\mathcal{V}_j \times \mathbf{S}_j$ to $\mathbf{S}_j$ via A_j, as well as projects the image $\mathcal{V}_j$ via π_j; via (π_j, A_j).

Moreover, let's say $\mathbf{S}_i$ observes its neighbourhood as embedding *Emb* and decides to make an action after satisfying the conditions; but at the same time $\mathbf{S}_j$ (a fiber of the vertex connect to the vertex of $\mathbf{S}_i$) also decided to make transition to other state space and eventually turned $|\mathbf{S}_j|$ less than the required number $\mathbb{N}$. So, locally for $\mathbf{S}_i$ satisfied the conditions but globally it is not possible due to concurrent choice of $\mathbf{S}_j$.

Notice that there are two ways to visualise a point in the state space (just as in fiber bundles); first, as a set whose elements consist of states which are projected to each vertex using π (state/fiber space); second, the same point has a dual characterisation as being part of the total space as discussed in the fiber bundle example 30. We construct the equivalent computational modelling of empirical models as base space modulo the fiber space. So, the cross product between $\mathbf{S}_i$'s and $\mathcal{V}_i$'s is in the total space. $\mathcal{V}_i \times \mathbf{S}_i$ and $\mathcal{V}_j \times \mathbf{S}_j$ are formed from different local trivialisation maps and is locally always possible to slice the total space in these thin lines (like imagine as thin perpendicular lines of specific thickness in case of cylinder of Figure 29 in Appendix A.1); but these different slices can be merged in the total space if there is a map δ between them. It means, the total space can be glued together as the base space agrees on the intersection, denoted as a dotted purple like in fiber bundle description in Figure 30.

Since $\mathcal{V}_i \times \mathbf{S}_i$ is an ordered pair in the total space; as well as implicitly if you consider a standard fiber $\mathbf{S}$ (as in fiber bundles), the action A_{ij} is $A_i : \mathbf{S}_i \to \mathbf{S}$ and similarly from other side $A_j : \mathbf{S}_j \to \mathbf{S}$; which means $\delta = A_j \cdot A_i^{-1}$ converting two different local trivialisation into one another, and further validating the action and the intersection globally. After, the conditions are checked, i.e., $|\mathbf{S}|$ and $w(\mathcal{K})$ which updates the vertices and their corresponding fibers for the next iteration.

Each iterations of TIM forms a 3-dimensional structure as shown in Figure 7; as well as possible infinite iterations, which looks like infinite stairs as shown in Figure 9.

An Overview of TIM: Let $\mathcal{E} \sim X$ be the topological environment with $\mathcal{U}_i$ and $\mathcal{U}_j$ as its open sets which consist of all the contexts *covering* $\mathcal{E}$; combinatorially described as a simplicial complex $\mathcal{K}$. The readers can refer to Figure 10 for building up the intuition of these concepts. When $\mathcal{U}_i \cap \mathcal{U}_j$ agrees at the overlap there is an edge between vertices of $\mathcal{K}$, which also contains a subset of simplices known as critical simplices CS where $\mathcal{U}_i \cap \mathcal{U}_j$ of $\mathcal{E}$ doesn't agree. Let $\mathcal{V}$ be the vertices of $\mathcal{K}$ embedded in $\mathcal{E}$, i.e., $\mathcal{K}$ is triangulation of $\mathcal{E}$. Each $\mathcal{V}$ of $\mathcal{K}$ is a hyper-space,(i.e., it has some additional information as in bigraph model or fat graphs in gauge field theory) consisting of a set of values representing possible states/behaviour realised as the state space $\mathbf{S}$. The state space $\mathbf{S}$ possess a set of states s associated to each $\mathcal{V}$ of $\mathcal{K}$. The map π assigns $\mathbf{S}$ to each $\mathcal{V}$, like, each $\mathbf{S}_i$, where i is any state space in $\mathbf{S}$, to each $\mathcal{V}_i$, where i is any vertex of $\mathcal{K}$. $\mathbf{S}$ can have different number of states at each $\mathcal{V}$. The inverse of π map of any state $s_i \in s$ contained in $\mathbf{S}_i$ is a particular vertex of $\mathcal{K}$, i.e., $\pi^{-}(s_i) = \mathcal{V}_i$. The embedding

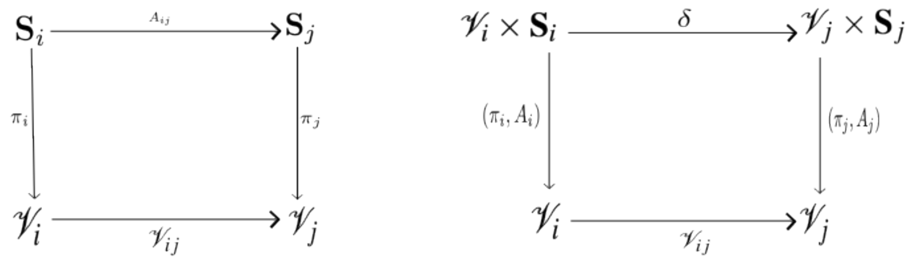

Figure 8: Pre-condition on left and Post-condition on right

Emb of π^{-1} of any state $s \in \mathbf{S}$ i.e., *Emb* $\{\pi^{-}(s_i)\}$ is the set of all simplices of $\mathcal{K}$ having projection of $\pi^{-}(s)$ (which is a vertex) as its element. An action $a_i \in A$ is admissible between any permutation of state s_i and s_j of $\mathbf{S}_i$ and $\mathbf{S}_j$ respectively if the condition *Emb* $\{\pi^{-}(s_i)\} \cap CS = \emptyset$ is satisfied. Action a_i happens in a given subset of simplices of $\mathcal{K}$ which represents the state of the environment. For simplicity, set a bound on the cardinality of state space to some natural number $\mathbb{N}$. A computational step $s_i \xrightarrow{a_i} s_j$ would reduce cardinality of state space of $\mathbf{S}_i$ containing s_i to one and $\mathbf{S}_j$ containing s_j gains one, similar to bio-inspired models of computation. If the cardinality $|\mathbf{S}_i|$ is less than a chosen bound then it turns the generic simplices to CS. The other map δ observes the effect of A on $\mathcal{K}$ and iterates the environment based on overall computations during that computational step. The iterated $\mathcal{K}$ is the new environment for the next computation.

Concisely, the computational process goes as: Suppose t is time and $\mathbb{N}$ is a natural number. At $t = start$, $|\mathbf{S}| > \mathbb{N}$ and $|CS| = \emptyset$. At $t = t + 1$, $s_i \xrightarrow{a_i} s_j$ for action $a_i \in A$. Check, if *Emb* $\{\pi^{-}(s_i)\} \cap CS = \emptyset$ then A is allowed, else not allowed. For a transition $s_i \to s_j$; delete s_i from $\mathbf{S_i}$ and add s_i to $\mathbf{S_j}$. At $t = t + 1$, if $|\mathbf{S}| < \mathbb{N}$ in $\mathbf{S_i}$ then turn simplex to CS. Iterate the simplicial complex using (generalised) DMT.

An Interlude There is a very important point to note here. The above example is an instance of interaction between computation and structure; but in a more general setting, a local computation being able to change the weight/colour of the simplex would affect the homotopy-type of the space, i.e., turning a generic simplex to critical simplex would affect all the simplices of $\mathcal{K}$ globally; because there is a rule governing the distribution of critical-generic simplices based on the theory and mathematical structures involved.

Let's say, we involve discrete Morse theory which defines a discrete Morse function, classifying simplices of $\mathcal{K}$ into: generic and critical. Now, changing any simplex to critical would require to consistently re-classify the space (even though the re-classification would require more mathematical extensions to the theory which is not relevant here) by re-setting all the simplices of the space based on discrete Morse function. The (re)classification would release a set of critical simplices to generic, as well as, turn a set of simplices to critical. Any change in the arbitrary far apart contexts, say, turning a simplex to critical, would change the homotopy type of the space. An instance of this general case was shown in the above example where $w(\mathcal{K})$ constrains action A and $|\mathbf{S}|$ in turn changes $w(\mathcal{K})$.

We do not intend here to formalise this *general* synergistic interaction between computation and structure because it is hard to formalise the machine where there is a possibility that any far apart arbitrary contexts could influence computation– *action at a distance.* It would become quite challenge to know the next state of the machine because the phase space is not known *a priori* unlike Turing-like interactive models. The *a priori* known phase space is foundational to theorise systems in natural sciences; otherwise a non pre-stated phase space leads to fundamental complexities [11], hence, a general theory to formalise this openness is not addressed.

Propitiously, what allows formalisation of structure-computation interaction for empirical models is their already known compactified tabular data. In fact the synergistic interaction between structure and computation emerges as a necessary additional structures to express contextuality in computation. Quantum contextuality is an empirical phenomenon arising from interaction between observer and microcosm. The observer discovers patterns in data of a physical experiment which is not known a priori in principle.

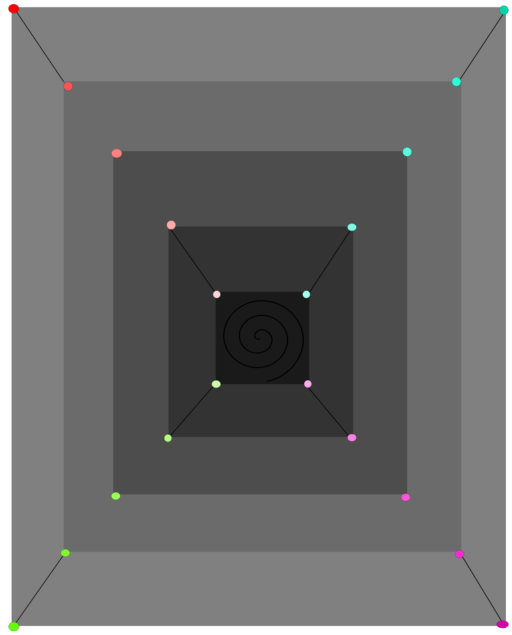

Figure 9: Representation of (infinite) computational steps based on the working of TIM as discussed in Subsection 3.2. We use coloured dots here to represent different symbols and avoid different maps between them for clarity of the image; green and red dots represent $\mathcal{V}_i$ and $\mathbf{S}_i$ related via π map; and pink and blue colour represent $\mathcal{V}_j$ and $\mathbf{S}_j$ related via another π map; which could go on infinitely enclosing the area of the paper; compare it with the single computational step in Figure 7 to further understand the meaning of different colours. The δ map updates $\mathcal{V}_i \to \mathcal{V}_i^{'}$,$\mathcal{V}_j \to \mathcal{V}_j^{'}$, $\mathbf{S}_i \to \mathbf{S}_i^{'}$ and $\mathbf{S}_j \to \mathbf{S}_j^{'}$; which also could go on infinitely enclosing the depth (3-D view) of the paper. The intensity of colours represent such updates; notice for instance, the red dot slowly fades its colour representing the updates that could go on infinitely represented as spiral.

Definition 2. *(**TIM**) A Topological interactive machine (see Figure 10) is a sextuple $(\mathcal{V}, \mathcal{K}, \mathbf{S}, \pi, A, \delta)$ where:*

- *$\mathcal{V} = \{\mathcal{V}_i\}_{i\in\mathbb{N}}$ is a finite (possibly unbounded) set of flexible state variables embedded in the topological environment $\mathcal{E}$ such that each vertex of its combinatorial space $\mathcal{K}$ is an element of $\mathcal{V}$.*
- *State space $\mathbf{S} = \{\mathbf{S}_i\}_{i\in\mathbb{N}}$ consisting of a finite set of processes each containing a finite set of states s.*
- *$\pi : \mathcal{V} \to \mathbf{S}$ assigns to each $\mathcal{V}_i$ a state space $\mathbf{S}_i$. Each element $s_i \in \mathbf{S}_i$ observes its corresponding simplex of $\mathcal{K}$; as well as its embedding Emb. Any action A between states of $\mathbf{S}_i$ and $\mathbf{S}_j$ is allowed only if Emb $\{\pi^{-}(s_i)\} \cap CS = \emptyset$ and $|\boldsymbol{S} > \mathbb{N}|$ such that the left square in Figure 8 commutes. It is the pre-condition expressing local consistency.*
- *A is the set of actions `A` defined as is a mapping function `A` $: \boldsymbol{S} \to 2^{\boldsymbol{S}}$ that maps each state $s \in \boldsymbol{S}$ to an element of the power set of $\boldsymbol{S}$. .*
- *$\delta : \mathcal{V} \times \boldsymbol{S} \to \mathcal{V} \times \boldsymbol{S}$ takes vertices in $\mathcal{K}$ and states in $\boldsymbol{S}$ and maps to another $\mathcal{K}$ and $\boldsymbol{S}$ such that the square on right side of the Figure 8 commutes. It is the post-condition expressing global consistency.*

$$\begin{array}{ccc} \boldsymbol{S}_i & \xrightarrow{A_{ij}} & \boldsymbol{S}_j \\ \downarrow{\scriptstyle \pi_i} & & \downarrow{\scriptstyle \pi_j} \\ \mathcal{V}_i & \xrightarrow{V_{ij}} & \mathcal{V}_j \end{array} \qquad \begin{array}{ccc} \mathcal{V}_i \times \boldsymbol{S}_i & \xrightarrow{\delta} & \mathcal{V}_j \times \boldsymbol{S}_j \\ \downarrow{\scriptstyle (\pi_i, A_i)} & & \downarrow{\scriptstyle (\pi_j, A_j)} \\ \mathcal{V}_i & \xrightarrow{V_{ij}} & \mathcal{V}_j \end{array}$$

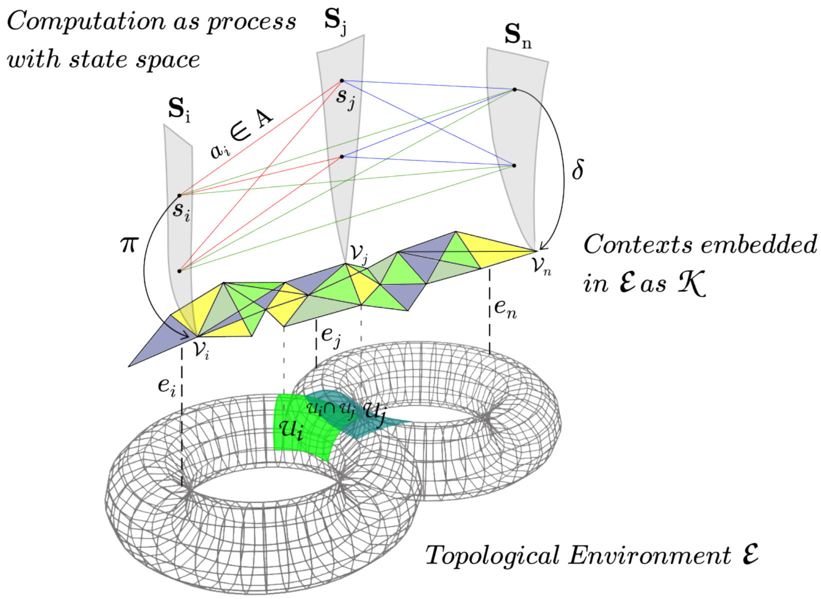

Figure 10: Topological Interactive Machine. The δ map looks same as π map; which is actually not the case but for visual simplicity we avoid over-representation of maps.

3.3 Topological Interactive Machine is a Computational Fiber Bundle

It is a natural observation that TIM is isomorphic to the structure of fiber bundle as seen in Figure 10. There is a conceptual point to note; in classical setting of bundle theory, the total space E is *known* and described in terms of base space B and fiber space F, like in Hopf fibration where 3-sphere as E is described in terms of circles as F and a sphere as B; unlike TIM where total space is *unknown* and we construct it via synergistic computation-structure interaction from the *known* base space ($\mathcal{K}$) and *known* fiber space ($\mathbf{S}$). The information about $\mathcal{K}$ and $\mathbf{S}$ is a priori known and extracted from the given information of the empirical models. Hence, E is not explicitly included in the definition of TIM because it is unknown and mainly expressed through interaction of different mappings in particular the update map δ. E in TIM is structure *modulo* computation where fundamental group of $\mathcal{K}$ encodes structural semantics and symmetry group encodes computational semantics in $\mathbf{S}$ which is extensively discussed in Section 3. This interplay iterates intermediate E's until all classes of behaviour from the empirical model are expressed by TIM.

This abstraction of E in the definition of TIM results in a few refinement without loosing equivalence with fiber bundles. Here is how E comes-in via interaction between different mappings: First, TIM defines π as $\pi : \mathcal{V} \to \mathbf{S}$ unlike fiber bundle where same map means $\pi : E \to B$. A computation $s_i \xrightarrow{a_{ij}} s_j$ for $s_i \in \mathbf{S}_i$, $s_j \in \mathbf{S}_j$ and $a_{ij} \in A$, takes place *locally* between two state spaces $\mathbf{S}_i$ and $\mathbf{S}_j$ given colour code of the edge $\mathcal{V}_{ij}$ between $\mathcal{V}_i$ and $\mathcal{V}_j$ in a local space; where it is always possible to locally describe each computational site as $\mathcal{V}_i \times \mathbf{S}_i$ and $\mathcal{V}_j \times \mathbf{S}_j$ via (π_i, θ_i) and (π_j, θ_j) respectively as shown in Figure 12. $\mathcal{V}_i \times \mathbf{S}_i$ and $\mathcal{V}_j \times \mathbf{S}_j$ are different local trivialisations residing in E which may not necessarily agree expressing the global consistency condition quantifiable via δ map as shown in Figure 11. The δ_i map in the local space is the simplicial map preserving fibers and updates $\mathcal{V}_i \to \mathcal{V}_i^{'}$ along-with $\mathbf{S}_i \to \mathbf{S}_i^{'}$, similarly for δ_j, based on conditions.

The above equivalent description of bundles operates in a changing base-fiber space in a computational (open) fiber bundle description as TIM which models empirical models.

Following the above points, we give an equivalence between bundles and TIM as in Figure 12. The right cube represent a complete iteration of a fiber bundle via different maps with slight abuse of notation 12 (repetitive maps are omitted and those which are not important for now). We have two open sets $\mathcal{U}_i$ and $\mathcal{U}_j$ whose pre-image is in the total space E as $\pi^{-1}(\mathcal{U}_i)$ and $\pi^{-1}(\mathcal{U}_j)$ respectively. It has two values; one is its projection (via π) onto $\mathcal{U}_i$ and other is the height which should be equal to a standard fiber F via map ψ. Informally, the strip should be isomorphic to the cartesian product of $\mathcal{U}_i \times F$ via (π, ψ); which is local trivialisation condition. These stripes glue together to form whole space. The process is similar for other open neighbourhood $\mathcal{U}_j$, which maps

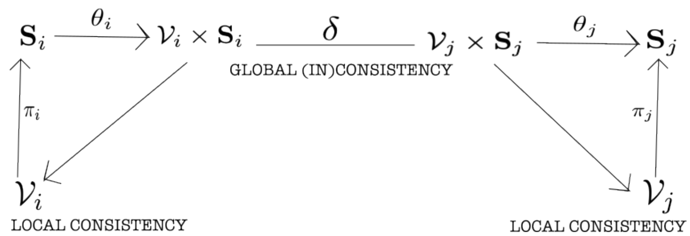

Figure 11: TIM's local consistency same as local trivialisation in bundles along-with its global consistency same as map between different local trivialisation. For further comparison refer to Figure 30.

to $\mathcal{U}_j \times F$ via (π, σ). Now, the intersection between $\mathcal{U}_i$ and $\mathcal{U}_j$ denoted as $\mathcal{U}_{ij}$ is compatible if there is a feasible map between $\mathcal{U}_i \times F \to \mathcal{U}_j \times F$ via δ map. The δ map is compatible with the intersection then the $\mathcal{U}_i \times F$ and $\mathcal{U}_j \times F$ will be projected back to the same $\mathcal{U}_i$ and $\mathcal{U}_j$ respectively, else different open set, say $\mathcal{U}_i^{'}$ and $\mathcal{U}_j^{'}$. For further understanding, you may refer to Figure 29.

The left cube represent the computation and update process in the topological interactive machine isomorphic to the fiber bundles. Here, $\mathcal{V}_{ij}$ has given weight w determining whether there is an edge between two vertices $\mathcal{V}_i$ and $\mathcal{V}_j$; represented as red or green using discrete Morse theory. One can follow same path as fiber bundle and δ determines whether the vertices $\mathcal{V}_i$ and $\mathcal{V}_j$ are projected to itself with same weight; or other vertices, say $\mathcal{V}_i^{'}$ and $\mathcal{V}_j^{'}$. The updating process via δ can be seen in the Figure 9. It should be noted that any π^{-1} map is an ordered pair; with one element from the base space and other element from fiber F. It should have been say $\mathcal{U}_i \times \pi^{-1}(\mathcal{U}_i)$ in case of right cube and $\mathcal{V}_i \times \mathbf{S}_i$ in left cube; but $\pi^{-1}(\mathcal{U}_i)$ or $\mathbf{S}_i$ should be isomorphic copies of standard fiber F or $\mathbf{S}$ respectively.

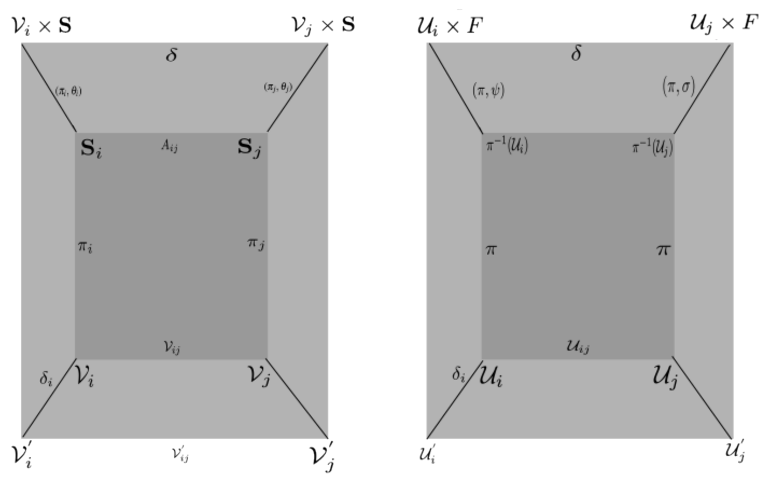

Figure 12: Topological Interactive machine isomorphic to fiber bundle structure. Note $\mathcal{V}_i \times \mathbf{S}_i$ and $\mathcal{V}_j \times \mathbf{S}_j$ is in total (global) space and same points would be viewed as $\mathbf{S}_i^{'}$ and $\mathbf{S}_j^{'}$ respectively in local space via updating process.

4 Empirical Model ≡ TIM

The mathematical semantics of TIM is based on the *fundamental theorem of the covering spaces* which provides an equivalence of categories between fundamental group(oid) of X and permutation of fibers of X in a (fiber) bundle setting.

The fundamental group at a point can be reconstructed as a group of deck transformations of the universal covering space, which is same as the automorphism of the fiber over that point of the projection map, i.e., the functor which sends a covering space of X to its permutation representation (monodromy set-action) of the fundamental group(oid) of X on the fibers of the covering space (total space) E is an equivalence of categories. The theorem is an instance of the general principle of Galois theory.

Equivalently, the mathematical structure of the structure-computation synergistic interaction in TIM has Galois-covering correspondence theorem in the background. The computation part of TIM represented as process is formalised as symmetry group and the topological environment is formalised as fundamental group.

Galois equality condition for contextual semantics It is natural that TIM can be viewed in the light of Galois theory. Here we give a general intuition. Suppose a finite topological environment $\mathcal{E}$ (hidden variable) gives to each vertex $\mathcal{V}_i$ a random environmental variable e_i. $\mathcal{V}_i$ is represented as a flexible state variable in TIM that is immersed in $\mathcal{E}$, realised as a combinatorial structure of a simplicial complex $\mathcal{K}$. There is a reason to distinguish between $\mathcal{K}$ and $\mathcal{E}$ because in general, the collapsibility of $\mathcal{K}$ does not imply contractibility of $\mathcal{E}$. The state space $\mathbf{S}$ consists of a possible set of states s_i that flexible variables hold. The composition of environmental variables and flexible variables form an equation, call it a *Galois equality*, that is satisfied by the $s_i \in \mathbf{S}$. The Galois equality is of the form, $\mathbf{p} = e_1\mathcal{V}_1 + e_2\mathcal{V}_2 + e_3\mathcal{V}_3 + ... + e_n\mathcal{V}_n = \mathcal{I}$ where set $\{e_1, e_2, ...e_n\} \in \mathcal{E}$ are hidden environmental variables and $\{\mathcal{V}_1, \mathcal{V}_2, ..., \mathcal{V}_n\}$ as flexible variables that correspond to vertices of $\mathcal{K}$ and $\mathcal{I}$ is the model-specific topological invariant. The solution or possible states s of each $\mathcal{V}$ in the equality are contained in $\mathbf{S}$, possibly a fiber over each $\mathcal{V}$ through projection map π of TIM. The automorphism of s in $\mathbf{S}$ is tied to $\mathcal{E}$, i.e., whatever happens in $\mathbf{S}$ equivalently affects topological environment $\mathcal{E}$. Equivalently, $e_1\mathcal{V}_1 + e_2\mathcal{V}_2 + e_3\mathcal{V}_3 + ... + e_n\mathcal{V}_n \to$ Symmetry group S_n and $\mathcal{I} \to$ fundamental group of topological environment $\pi(\mathcal{E})$ of the above equality. The idea of existence of global sections and solution of linear systems has been used by Abramsky [5].

Violation of Galois equality implies contextual semantics in TIM, because the non-solvability of Galois equality mean non-trivial (co)homology of X, implying obstructions and hence non-locality and contextuality in TIM. A connected topological space invariant under symmetry group can't change under any permutation unless going through non-trivial region. In this sense, contractible/collapsible (discrete) spaces retain symmetry preserving locality/non-contextuality. The existence of the non-trivial region (holes) in topological space obstructs this symmetry; and in the synergistic computation-structure interaction, this obstruction is quantifiable as a *symmetry breaking process*, meaning that the topology could change due to arbitrary a far contexts making a local computation, which captures the essence of contextual semantics machinery. It could be physically compared with spontaneous/dynamical symmetry breaking in gauge theory which are also structured in fiber bundle formalism.

As a result, the computation part of TIM is expressed by symmetry group which considers all the possible permutation of states in the state space. The structure part of TIM is expressed by the fundamental group quantifying paths on a discrete topological space, here simplicial complex, using DMT. Both groups acts together as a unique algebraic object as in case of fiber bundles. The structure group acting on a principle fiber bundle is called a gauge group. It is a semi-direct product of symmetry group and fundamental group. Gauge group is the measure of contextual semantics in TIM.

Proposition 1. *A Topological interactive machine emulates behaviour bisimilarity of the empirical model* **e** *isomorphic to the structure of fiber bundle.*

Proof. Empirical model structures as sheaves as shown in Subsection 2.2 and extensively discussed in [5, 8]. Also, there exists an adjoint functor between the category of sheaves and the category of fiber bundles on a given topological space [19] [In particular, Theorem 2 of Section 5 of Chapter II]. Moreover, TIM is isomorphic to the structure of fiber bundle by Definition 2 and extensively discussed in Subsection 3.3; which implies that TIM shows behavioural bisimilarity of the empirical model. The contextual behaviour of the empirical model is quantified as a gauge group which encodes TIM computational machinery expressing the class of behaviour and strength of contextuality of empirical models; for instance logical contextuality in Hardy model and strong contextuality in Kochen Specker model as discussed in Section 4 □

Explanation The sheaf of sets $\mathcal{O}$ over a topological space X is locally homeomorphic to X unlike in fiber bundle the space is not locally homeomorphic with base space X, but locally homeomorphic to $\mathcal{U} \times F$, for suitable open subset $\mathcal{U} \subseteq X$. So, naively one needs to weaken the condition in the fiber bundles and restricting it to local homeomorphism to $\mathcal{U}$ turning into sheaves. Conversely, adding to sheaf of sections and defining a topology on it would extend its local homeomorphism turning it to a fiber bundle. Infact, there is an equivalence between a particular bundle and sheaves. For instance, several theorems in algebraic geometry are on correspondence between vector bundles and particular sheaves; also the correspondence between étale bundles and sheaves. Every sheaf comes from bundles and conversely every sheaf gives rise to a bundle in an informal sense. Infact, the set of local sections of fiber bundle form sheaf over the topological space. Every sheaf is a sheaf of cross-sections. Here, topological obstructions quantify non-locality and contextuality which allows this peculiar structural equivalence between bundles and sheaves to be sufficient for equivalent ways to measure this feature.

A background relatability The non-vanishing connecting homomorphism in homological sequence is responsible for the obstruction to the existence of global sections quantifying non-locality and contextuality []. Every homology theory that satisfy the homology axioms must give a long exact sequence for every pair of space, say $(X, \mathcal{A})$ where $\mathcal{A}$ is an abelian group, which involves relative homology groups, i.e., $\cdots \to H_n(\mathcal{A}) \to H_n(X) \to H_n(X, \mathcal{A}) \xrightarrow{\gamma} H_{n-1}(\mathcal{A}) \to \cdots$, where $\xrightarrow{\gamma}$ represent mapping between different homology group like from H_n to H_{n-1} and γ is their connecting homomorphism. We can however turn it into sequence of absolute homology groups, i.e, $\cdots \to \tilde{H}_n(\mathcal{A}) \to \tilde{H}_n(X) \to \tilde{H}_n(X/\mathcal{A}) \xrightarrow{\gamma} \tilde{H}_{n-1}(X) \cdots$, using excision theorem to relate the relative homology group with the reduced homology group of the quotient, $H_n(X, \mathcal{A}) \sim \tilde{H}_n(X/\mathcal{A})$. So, the short sequence of spaces $0 \to \mathcal{A} \xrightarrow{m} X \xrightarrow{n} X/\mathcal{A} \to 0$ leads to long exact spaces in homology. The short exact sequence can split if there exists a homomorphism $h : X/\mathcal{A} \to X$ such that the composition $n \cdot h$ is the identity map on $X/\mathcal{A}$ which gives it the structure of direct sum, i.e., $B \sim \mathcal{A} \times C$ where $C = X/\mathcal{A}$. This factorability due to the fact that connecting homomorphism γ in long exact sequence (by applying snake lemma) at a specific context C and corresponding section say r is equal to zero, i.e., $\gamma_C(r) = 0$; there exist a compatible family such that the local sections can be extended globally, otherwise not extendible meaning non-locality and contextuality.

Similarly, homotopy is a non-local property of spaces and has been used to quantify non-locality and contextuality []. The excision theorem cannot yield long exact sequence of absolute homotopy group, but instead one can look at quotient maps satisfying homotopy lifting property yielding long exact sequence in fiber bundles, thus a potential candidate to quantify homotopy for measuring non-locality and contextuality. Moreover, several results in this context (like, Serre spectral sequence and Blaker's Massey theorems) provide vanishing criteria structured over fiber bundles to compute (co)homology and homotopy group of the total space from the information of base space and fibers and vice-versa. The examples of models in Section 4 has a possibility to be viewed (slight-differently) as pages of (serre) spectral sequence (generalisation of exact sequences); where at each iteration, for instance in Hardy model as in Subsection 6.2, the first iteration outputs a torus which corresponds to first page of spectral sequence which eventually expresses all class of behaviour as 3-torus corresponding to third page of the sequence.

A guiding idea of bundle realisation of Bell scenario Bell scenario consisting of agents $\mathscr{N}$. Each agent has a choice of different measurements to perform from a measurement set $\{m_1, \ldots.., m_n\} \in \mathscr{M}$; which yields possible outputs $\{o_1, \ldots..o_n\} \in \mathscr{S}$ characterised by triplet $(\mathscr{N}, \mathscr{M}, \mathscr{S})$ with a probability. The joint probability of obtaining the outcomes $\{o_1, \ldots..o_n\}$ given the measurement settings $\{m_1, \ldots.., m_n\}$ will be denoted by $p_{o_1 \ldots o_n}|_{m_1 \ldots m_n}$. These probabilities through all agents traversing all measurement-output permutations represented as $t = \prod_{i=1}^{\mathscr{N}} (\sum_{j=1}^{\mathscr{M}_i} \mathscr{S}_i^j)$ forms components of a vector $\vec{p}$ in affine space $\mathbb{R}^t$ known as behaviour space. A behaviour can be viewed as a point $p \in \mathbb{R}^t$ defined by the positivity and normalisation constraints which gives it a topology realised as simplicial complex. Each permutation of $(\mathscr{N}, \mathscr{M}, \mathscr{S})$ corresponds to specific region in the behaviour (topological) space either classical, quantum or no-signalling region. Any permutation of triplet has an associated region in the topological behavioural space.

Both triplet and topology can be associated naturally as a fiber bundle; where the scenario is represented as a simplicial complex and the fibers being the set of outcomes, and contextuality as non-existence of global section in the bundle description. The guiding intuition is to consider each agent as a hyper-vertex, inside which reside measurement set $\{m_1, \ldots.., m_n\}$ represented as vertices. To each $m_i \in \mathcal{M}$, assign all values of the output set $\mathcal{S}$, i.e., for each agent, to each m_i associate set $\mathbf{o} = \{o_1, \ldots..o_n\}$ as a fiber to each vertex. The computation for each measurement m_i yielding all possible outputs $\mathbf{o}$, i.e., $m_i \to \mathbf{o}$ at i^{th} agent, is represented as a point $\mathbf{p}_i$ in the behavioural space. These points can be structured as simplicial complex which represents the probability distribution whose vertices describe the computation $m_i \to \mathbf{o}$. As a result, one can embed each vertex representing measurements m_i associted with output set $\mathbf{o}$ as fiber into the simplicial complex giving it a fiber bundle description.

Symmetry group encode PTM computation

We construct a $\mathcal{K}$ associated with each model in Section 4, for instance, a discrete 3-torus is associated with the Hardy model as in Figure 22. Each vertex of $\mathcal{K}$ (3-torus) have an associated state space (fiber) $\mathbf{S}$. We are interested in class of irreducible behaviour of the state space so that we do not have to consider each fiber associated to same class. All the possibilities of computation in the state space can be encoded as different permutation in symmetry group S_n (like S_4 in the Hardy model), so we look at irreducible representation of S_n to encode the irreducible behaviour in computational space.

A representation of a finite group $\mathcal{F}$ over field $\mathbb{F}$ is a homomorphism $\rho : \mathcal{F} \to GL(V)(= Aut(V))$ where V is a vector space of finite dimension over $\mathbb{F}$. $GL(V)$ is general linear group and $Aut(V)$ is the automorphism group of V, i.e., group of linear invertible transformations $T : V \to V = GL(V)$. The vector space V is called the representation space of $\mathcal{F}$. The representation theory gives a way to associate every element of group to matrices and group operation to matrix operation. The vector space is an abelian group under addition as is the state space under transformation/function as permutation group. The group-theoretic representation of regular languages is expressed as automaton which is central to formal language theory and a way to see vector space as state space.

Remark. *There exists no invertible transformation T between irreducible representation of different dimensions.*

Theorem 1. *Let $\mathcal{F}$ be a finite group, $\mathbf{D}_i$ and $\mathbf{D}_j$ be irreducible representations of state space $\mathbf{S}_i$ and $\mathbf{S}_j$ over state set s respectively. There exists an admissible linear transformation (computation) $\mathbf{T}$ from $\mathbf{S}_i$ to $\mathbf{S}_j$ i.e., transition $\mathbf{T} : \mathbf{S}_i \to \mathbf{S}_j$ if and only if $\mathbf{T} \cdot \mathbf{D}_i(g) = \mathbf{D}_j(g) \cdot \mathbf{T}$ for all $g \in \mathcal{F}$, else it is a forbidden transition or contextual computation.*

Proof. Schur lemma deals with the equivalence of irreducible representations. For a finite group $\mathcal{F}$; and $\mathbf{D}_i$ and $\mathbf{D}_j$ be irreducible representations of state space $\mathbf{S}_i$ and $\mathbf{S}_j$ over s. Suppose we are able to construct a linear transformation $\mathbf{T} : \mathbf{S}_i \to \mathbf{S}_j$ such that $\mathbf{T} \cdot \mathbf{D}_i(g) = \mathbf{D}_j(g) \cdot \mathbf{T}$ for all $g \in \mathcal{F}$ condition holds i.e., $\mathbf{T}$ intertwines the two irreducible representations then there are three possibilities as per Schur criteria.

1. $\mathbf{T} = 0$ then $\mathbf{D}_i$ and $\mathbf{D}_j$ are not equivalent and transitions are forbidden like deadlocks.
2. $\mathbf{T} \neq 0$ and $\mathbf{T}$ is a singular transformation then $\mathbf{D}_i$ and $\mathbf{D}_j$ are not equivalent and transition responsible for locally consistent and globally inconsistent computation.
3. $\mathbf{T} \neq 0$ and $\mathbf{T}$ is a non-singular (invertible) transformation, $\mathbf{D}_i$ and $\mathbf{D}_j$ are equivalent and transition is admissible.

□

The local transformations $\mathbf{T}$ between $\mathbf{S}$ are carried out in the presence of underlying simplices of $\mathcal{K}$. The singular transformation over $\mathbf{S}$ is locally feasible but globally infeasible due to the presence of non-trivial loops in $\mathcal{K}$. So, these irreducible representations of the symmetry group are subjected to structural constraint of the discrete structure which would decide whether there exits an intertwining transformation between them as shown in the fiber space description of various empirical models as in Kochen-Specker 27, Greenberger-Horne-Zeilinger model 26, PR box 24 and Hardy model 21. Notice, in models like Kochen-Specker, PR box and Greenberger-Horne-Zeilinger there exists no such $\mathbf{T}$ because of structural constrain of their associated discrete manifold on their corresponding irreducible representations of symmetry groups.

Fundamental group encode topological environment

Theorem 2. *The non-trivial fundamental group of the topological environment $\mathcal{E}$ implies contextual semantics in TIM computational framework.*

Proof. $\pi(\mathcal{E}) = 0$ corresponds to simply-connected space. The simply connected spaces are product space in which every local section can be extended to global section by the definition of TIM. The local sections in non-trivial spaces cannot be globally extended: $LC - GI$, which means the fundamental group of $\mathcal{E}$ is non-trivial. It infers non-vanishing homotopy groups. $\mathcal{K}$ quantifies $\mathcal{E}$ as a set of paths in discrete setting using DMT as discussed below. □

5 Strong Collapsibility and Contextual Semantics

The Theorem 2 gives a general description about contextual semantics of TIM. The machine is not based on $\mathcal{E}$ but its equivalent discrete counterpart $\mathcal{K}$. The combinatorial structure of $\mathcal{E} \sim \mathcal{K}$ representing $\mathbf{e}$ encodes information about its contextual semantics as homotopy classes. The homotopy information of $\mathcal{K}$ is quantified using DMT.

The equivalent expressiveness of explicit topological environment encoding contextual semantics is based on strong collapse of its directed simplicial complex $\vec{\mathcal{K}}$, constraining the feasibility of computations over **S** unlike PTM. There is a relation between collapsibility of a complex and contextuality. The fundamental group is expressed in discrete setting as set of paths over simplicial complex using DMT which constrain transformation/computation of Theorem 4. It provides homotopy information about the space which is a measure of contextuality. We introduce concept of strong collapses and their relation with contextual semantics.

Strong collapsibility $\implies$ Locality / Non-contextuality The topological environment is combinatorially represented as simplicial complex and its collapsibility is characterised with contextual semantics of **e**. $\mathcal{K}$ is immersed in $\mathcal{E}$ and collapsibility of $\mathcal{K}$ doesn't infer contractibility of $\mathcal{E}$. There exist finite spaces that are not homotopy equivalent to their associated complexes. We introduce *strong collapses* such that the contractibility of topological spaces is equivalent to strong collapsibility of ordered simplicial complex $\vec{\mathcal{K}}$. The direction to $\mathcal{K}$ is given via discrete Morse function $\mathfrak{f}$ of DMT and classifies its vertices and edges as critical and generic simplices. Forman's beautiful paper is the standard reference on this subject [17]. A very short account on DMT is also provided in Appendix B. Moreover, the paths are classified as non-closed paths (gradient) and non-trivial closed paths. The possibility of non-trivial closed paths arises due to equidistant critical points as shown in the examples of the empirical models in Section 6. Strong collapses infer no critical simplices and non-trivial closed paths in $\vec{\mathcal{K}}$ which imply the existence of global section of **e** and factorability of topological environment. These non trivial closed paths are the virtual loops that characterise an essential topological transition and do not violate the definition of $\mathfrak{f}$ and causality as discussed in Subsection 5.1.

Definition 3. *Let* **e** *be an empirical model and $\vec{\mathcal{K}}$ its associated directed simplicial complex over discrete Morse function $\mathfrak{f}$. The $\mathtt{n}$ non-possibilistic events of* **e** *are represented by $\mathtt{m}$ critical simplices $\mathcal{CS}$ where $\mathtt{m} \geq \mathtt{n}$ and incompatible family of sections by class of non-trivial closed paths $\mathcal{CP}$ (see Figure 13).*

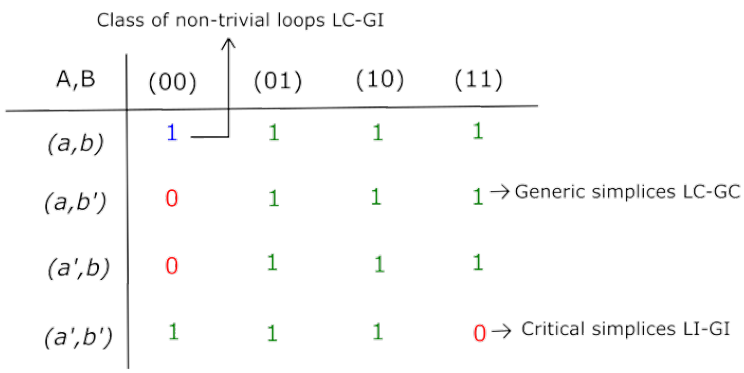

Figure 13: Topological characterisation of Hardy Table.

Definition 4. *Let $\vec{\mathcal{K}}$ and $\vec{\mathcal{L}}$ be directed simplicial complexes. We say that there is a* ***strong collapse*** *from $\vec{\mathcal{K}}$ to $\vec{\mathcal{L}}$ (indicated with $\vec{\mathcal{K}} \searrow \vec{\mathcal{L}}$) if there are no critical simplices $\mathcal{CS}$ and non-trivial closed paths $\mathcal{CP}$ from $\vec{\mathcal{K}}$ to $\vec{\mathcal{L}}$.*

If there exist no critical simplices in an interval say $[a, b]$ then it is collapsible to a null vertex $\{\emptyset\}$. But it is not the sufficient condition for contractibility of its associated $\mathcal{E}$. The obstruction to the existence of global section are holes in topological space that correspond to $\mathcal{CS}$. These induce cyclicity in simplicial complex and are responsible for its non-collapsibility up to homotopy. The non-existence of $(\mathcal{CS}, \mathcal{CS})$ signifies collapsibility of $\mathcal{K}$. It infers that corresponding empirical model has no non-possibilistic events and incompatible families of paths. These are *strong collapses* of $\vec{\mathcal{K}}$. The cyclicity in $\mathcal{K}$ is allowed in open environment of TIM computational framework.

Lemma 1. *If a directed simplicial complex $\vec{\mathcal{K}}$ is strong collapsible then its associated finite topological space $X \sim \mathcal{E}$ is contractible.*

Proof. Strong collapses imply no $\mathcal{CS}$ and $\mathcal{CP}$ in $\vec{\mathcal{K}}$. The non-existence of $\mathcal{CS}$ signifies collapsibility [17]. So we need to only prove relation of $\mathcal{CP}$ and collapsibility. Suppose we have discrete Morse function $\mathfrak{f}$ and a gradient path

$$\sigma_0^{(d)}, \tau_0^{(d+1)}, \sigma_1^{(d)}, \tau_1^{(d+1)}, \sigma_2^{(d)}, ..., \tau_r^{(d+1)}, \sigma_{r+1}^{(d)}$$

such that for each $i = 0, ..., r$, $\{\sigma, \tau\} \in V$ and $\tau_i > \sigma_{i+1} \neq \sigma_i$. Let this sequence be a *non-trivial closed path*, meaning that $r \geq 0$ and $\sigma_0 = \sigma_{r+1}$. Then, by definition of function of DMT we have $\mathsf{f}(\sigma_0^{(d)}) \geq (\tau_0^{(d+1)}) > (\sigma_1^{(d)}) \geq (\tau_1^{(d+1)}) > ....(\sigma_r^{(d)}) \geq (\tau_r^{(d+1)}) > (\sigma_{r+1}^{(d)}) = \sigma_0^{(d)})$ which is a contradiction. It could be only possible when we allow a transformation of fundamental group of X changing $\mathcal{E}$ to new topological environment $\mathcal{E}'$ which facilitates openness in TIM. □

Theorem 3. *If a directed simplicial complex $\vec{\mathcal{K}}$ is not strong collapsible then $\vec{\mathcal{K}}$ is contextual in nature.*

Proof. From Proposition 1, the empirical model is represented as $\mathcal{E}$ which is part of a fiber bundle description. Non-contractibility of $\mathcal{E}$ infers non-locality and contextuality. Under *strong* collapses, contractibility of $\mathcal{E}$ is equivalent to collapsibility of $\vec{\mathcal{K}}$. The acyclicity of $\mathcal{K}$ is the extendibility of every compatible family to a global section. Lemma 1 defines strong condition for $\vec{\mathcal{K}}$ to be strong collapsible. The existence of $(\mathcal{CS}, \mathcal{CP})$ in $\vec{K}$ is non-collapsibility to $\{\emptyset\}$. It infers existence of obstructions to global section as $\mathcal{CS}$ and induces incompatible family of paths as $\mathcal{CP}$. The corresponding cyclic Hasse diagram of $\vec{\mathcal{K}}$ infers contextuality.

□

Lemma 2. *Let* $\mathbf{S}_i \overset{\phi}{\rightarrowtail\twoheadrightarrow} \mathbf{S}_j$ *be permutations between state spaces* $\mathbf{S}_i$ *and* $\mathbf{S}_j$*, and* $\mathcal{V}_i$ *and* $\mathcal{V}_j$ *be two vertices of a simplicial complex* $\mathcal{K}$*. If the square*

$$\begin{array}{ccc} \mathbf{S}_i & \overset{\phi}{\dashrightarrow} & \mathbf{S}_j \\ \eta\uparrow & & \mu\uparrow \\ \mathcal{V}_i & \xrightarrow{\psi} & \mathcal{V}_j \end{array}$$

commutes i.e. $\eta \cdot \phi = \psi \cdot \mu = e$ *then the associated* $\mathcal{K}$ *of the model is collapsible and the global section exists.*

Gauge group encodes contextual semantics

Lemma 3. *The gauge group* $\mathcal{G}$ *of fiber bundle encodes contextual semantics as semi-direct product of symmetry group* S_n *of state space and fundamental group of discrete topological space* $\pi(\mathcal{K})$ *of TIM, i.e.,* $\mathcal{G} = \mathrm{S}_n \rtimes \pi(\mathcal{K})$*.*

Proof. By definition, the gauge group $\mathcal{G}$ is the structure group of the TIM. It has two components: symmetry group acting on PTM computation and fundamental group of $\mathcal{E}$ quantified in terms of paths over $\mathcal{K}$ using DMT. The interplay between both groups is described extensively in Section 3 and Section 4 respectively. So, $\mathcal{G} = \mathrm{S}_n \rtimes \pi(\mathcal{K})$. □

Remark. *The fiber bundle doesn't always admit a product topology, i.e., it cannot be always expressed as direct product of* $\mathcal{U} \times F$*. The direct product* $\times$ *generalises to the semi-direct product* $\rtimes$*. A computation* $s_i \xrightarrow{a_i} s_j$ *over state space is subjected to* $\mathcal{V}_i \to \mathcal{V}_j$ *over* $\mathcal{K}$ *in TIM framework. The effective computation cannot always be associated with linear environment unlike Turing computation. It is the computational meaning of the semi-direct product in mathematical sense.*

5.1 Virtual loops

DMT classifies simplices of $\mathcal{K}$ as generic and critical which correspond to feasible (locally consistent and globally consistent) and infeasible (locally inconsistent and globally inconsistent) computation respectively. The $LC - GI$ correspond to transitory virtual loops (non-trivial loops) that emerge in $\mathcal{K}$ due to equidistant critical point in the discrete topological spaces of the empirical models denoted in blue colour in Section 4. It does not violate the description of discrete Morse function f (loop-free space) and causality because these virtual loops vanishes when one reiterates f over final $\mathcal{K}'$ which had been constructed after going through several $\mathcal{K}$'s. In example of the Hardy model, the final $\mathcal{K}_4$ is reproduced after going through intermediate $\mathcal{K}_1$, $\mathcal{K}_2$ and $\mathcal{K}_3$, based on the information of its associated table discussed in Section 4; like union of discrete surface based on Morse theory for bundles [10], which gives a way to visualise the 3-torus structure of the Hardy model as union of three 1-torus. One can reiterate f on the structure at each level of iteration to yield a virtual-loop free discrete manifold. The transitory emergence of virtual loops in the discrete space quantifies the change in its homotopy class. As in same example, $\mathcal{K}_3$ is homeomorphic to 2-torus which iterates at next step to final $\mathcal{K}_4$ homeomorphic to 3-torus based on Hardy table. The transition from 2-torus to 3-torus could be viewed as a semi-stable state of the discrete manifold because in this very instant these virtual loops emerge due to equidistant critical points changing the homotopy class of the space, here from 2-torus to 3-torus. The equivalent mathematical description in dynamical systems of these flow categories are gradient flow, curl flow and harmonic flow corresponding to $LC - GC$, $LI - GI$ and $LC - GI$ based on de Rham cohomology via Hodge decomposition.

Conceptually, an experimenter does not have a priori access to the final discrete space but it collects data at each round of the experiment to construct the intermediate spaces. The simplicial complex is constructed from the observational data output from a Bell-like experiment, where there is an effective interaction between the observer and the microcosm. The total structure of the space is not a priori accessible to the observer and its discovery through an act of observing reproducing intermediate structures is a conceptual way to look at these virtual loops. The iterations of passing through intermediate discrete spaces of different homotopy class quantifiable as abstract-virtual loops could be seen as a symmetry breaking process.

6 Contextual Semantics Machinery: Examples

This section applies results of Section 3 to the examples of empirical models in the foundations of quantum physics, generalising these models in a computational way through TIM framework. TIM is structurally equivalent to these models expressing contextual semantics along-with the strengths of contextuality in different models, like logical contextuality in Hardy model and strong contextuality in Kochen Specker model. The only information given is the possibilistic table of the models. We provide examples of Hardy model, Kochen Specker model, Mermin-Peres Magic Square, Popescu-Rohrlich Boxes and Greenberger-Horne-Zeilinger Model, and the tables are given in the Appendix E for further information that might be relevant for the results of this section.

The idea is to construct the total space associated with empirical models from base space modulo fiber space. We prefer to represent them as semi-direct product of fundamental group and symmetry group reflecting computation-structure interaction. A different and equivalent algebraic description is evaluation of the irreducible representation of symmetry group subject to a discrete structure.

6.1 Computational Generalisation of Empirical Models

The Overview We are given the possibilistic tables of the empirical models. Lets take Hardy table to illustrate and it would be same for other models. We proceed with an overview of the idea and our focus is expressiveness of class of behaviour associated with these models as TIM.

Discrete Topological Space First, from the context set $\{a, b, a^{'}, b^{'}\}$, we construct a simplicial complex, say $\mathcal{K}_1$ by considering its power set which has a discrete topology as in Figure 15. $\mathcal{K}_1$ is the combinatorial description of all possible ways in which the basic set-up of the table can be performed. Notice, all simplices of $\mathcal{K}_1$ are feasible; one could think to randomly pick any non-possibilistic context say $ab^{'}$ at (00) from its possibilistic table ((00) in fiber space which is not considered yet) and mark it red over $\mathcal{K}_1$. But it turns out that there arises a context indistinguishability issue, i.e., there are six $ab^{'}$ over $\mathcal{K}_1$ so it would be obsolete to turn any $ab^{'}$ red or green randomly without any background condition. Moreover, choosing any context randomly would force fiber space to contain only its corresponding output, here (00), giving no element of choice to choose any values from output set, i.e., if one makes an edge as red between $ab^{'}$ over $\mathcal{K}_1$, it would mean that each corresponding fiber would consist only 0 (i.e., at (00)), because at (11) the given context is possible.

We use discrete Morse theory as a rule or a background condition in order to turn the simplices of $\mathcal{K}_1$ red that would represent non-possibilistic context of the table. The theory provides a systematic way to turn a set of particular context critical (red) based on discrete Morse function, hence the discrete structure puts constraint on the maximum number of critical simplices to be included on it. $\mathcal{K}_1$ is homeomorphic to topology of a real line or a disc because its Euler characteristics turns out to be 1. Correspondingly, $\mathcal{K}_1$ could contain a vertex and an edge, say a particular $ab^{'}$ among similar six $ab^{'}$ as critical using discrete Morse function. The cell complex formed from a vertex and an edge is homeomorphic to a disc via a proper attaching map. Also, the (co)homology of disc is trivial and contractible to a point and does not give any significant class of behaviour expressed in the table. The linear topology recalls tape structure of the Turing machine characterising local hidden variables which cannot reproduce contextuality due to Bell-Kochen-Specker theorem. The semantic of the structure, here $\mathcal{K}_1$ is described by the fundamental group which is quantified as loops in $\mathcal{K}_1$. One can easily compute the randomly selected particular context among other indistinguishable contexts by evaluating loops over $\mathcal{K}_1$ as described extensively in 6.2. The idea is to explore all possible loops, paths starting and ending at the same vertex, through each level (set-up) of the possible combinatorial set-up $\mathcal{K}_1$, and evaluate particular context at a particular level of the set-up which is not contained in the loop-set, would be the critical simplex. But DMT does not allow loops and one of the natural way is the identification of the boundary topologising $\mathcal{K}_1$from a trivial disc or real line topology to a surface, turning $\mathcal{K}_1$ to $\mathcal{K}_2$ as in the Figure 16.

Discrete Morse function assigns to each simplices a natural number $\mathbb{N}$ under the condition that as we move from lower to higher simplices, the number should increase. The identified simplices are assigned same value which puts constraint on the assignment of values to other simplices, and as a result there remain a set of simplices where the number decreases and turned red (critical) over $\mathcal{K}_2$ as shown in Figure 17. It turns that four 0-simplices, two 1-simplices and one 2-simplex turned red based on discrete Morse function as in Figure 17. The assignment of$\mathbb{N}$ to each simplices of $\mathcal{K}_1$ is not shown for brevity as well as it is a well known formalism. It is quite evident that $\mathcal{K}_2$ is homeomorphic to 1-torus, because if one evaluates the Euler characteristic of $\mathcal{K}_2$ it turns out to be zero, which is same as 1-torus; as well as there is an CW attachment map between red simplices which are homeomorphic to 1-torus. When we compare the red simplices to the table, we find that only two non-possibilistic contexts $ab^{'}$ and $a^{'}b^{'}$ are turned red. We would require to iterate another surface with $\mathcal{K}_2$ to include other class of behaviour of the Hardy table. The idea is to construct the topological space iteratively so that it can express all class of behaviour of the table; as well as at every iteration, using discrete Morse function we turn simplices red based on rule of the theory. $\mathcal{K}_2$ as 1-torus acts as a base space which will evolve through TIM machinery. The instances of different $\mathcal{K}$'s represent the evolution of $\mathcal{K}_2$ which in case of the Hardy model starts with 1-torus and evolves into 3-torus to express class of behaviour of the Hardy table.

TIM has synergy between the fiber space and the discrete topological space; so any simplex turned red could be invalidated because in irreducible fiber space those simplices might not be considered; as well as those which are not in the table. It is similar to post and pre condition of TIM changing the weights of simplices. So the total space associated with the Hardy model will be the quotient space of given topological space modulo the fiber space. We start another iteration which takes other non-possibilistic event $a^{'}b$ as shown in Figure 18. Similarly, as a check its Euler characteristic is -2, the 2-torus; which can be independently also constructed from the CW mapping between given n-simplices. All the non-possibilistic events are expressed on the discrete space which is homeomorphic to the 2-torus except the strange context (ab) which is $LC-GI$. The weaving of contexts is very peculiar, i.e., the construction of surface invokes similar iterations even after trying different permutations of the contexts, with an intention to see if we can get minimum surface to express all classes of behaviour; discrete Morse theory gives discrete manifold with maximum number of holes due to the fact that critical simplices are greater than the Betti number of the surface.

The context (a, b) cannot be expressed as a critical simplex because it should be possible when it observes its corresponding simplices via π map. The context (a, b) is expressed as (short-lived) virtual loop. So, we further iterate to $\mathcal{K}_3$ to $\mathcal{K}_4$ as shown in Figure 19. These non-trivial virtual loops emerge due to equidistant critical points. The edge joining vertices $a^{'}$ and $a^{'}$ (marked with a red arrow over their edge) has a choice to go in either directions; which facilitates non-trivial loops marked in blue. The virtual loops do not violate the description of discrete Morse function (loop-free space) and causality because these loops vanishes when one reiterates the function over final simplicial complex, here $\mathcal{K}_4$, which had been constructed after going through several $\mathcal{K}$'s. The virtual loops quantify a topological transition from one homotopy-class topological space to another. For instance, $\mathcal{K}_3$ realised as 2-torus changes to $\mathcal{K}_4$ realised as 3-torus; both homotopy inequivalent space. The Hardy model would have started as simply connected space like a disc (for the observer conducting the experiment) which after encountering a virtual loop changes the homotopy-type of space, turning to 1-torus and afterwards 2-torus and 3-torus, expressing all class of behaviours, where each iteration is a symmetry breaking process. This whole process can be described as a structure-computation synergistic interaction of TIM whose instance is illustrated in the example 3.2. LC-GI would mean an edge between (ab) at (00) is possible in the state space because there is no corresponding critical simplices but due to virtual loop in $\mathcal{K}_4$ it is not possible globally because it indicates the global topological transition from one homotopy class to another. As a result, the final space is homeomorphic to a 3-torus, which will be our base space of the Hardy model. It is measured by its fundamental group quantified via discrete Morse theory in discrete setting as in Theorem 5.

State Space The fiber space consists of the total possible outcomes represented as computation between different states as in Figure 10. In the Hardy case, there are four state spaces with permutations between their respective state; which would also be equal to the number of vertices (because π assigns to each vertex a fiber space). So, the measurement set (or respective fibers) in Hardy model are four, $\{a, b, a^{'}, b^{'}\}$ represented as a regular tetrahedron. Each symmetry of the tetrahedron corresponds to a permutation of its states in the state space (0 and 1 in each fiber). It accounts all the 24 symmetries of the tetrahedron along different planes. The group of all self-isometries that sends a regular tetrahedron to itself is isomorphic to S_4 symmetry group. One can imagine fiber associated to each vertex of its base space homeomorphic to 3-torus; but we are interested in its irreducible class of behaviour, because other behaviour would categorise into the irreducible ones. After this, we follow the standard mathematics of representation theory of symmetry groups as in Theorem 4 but subject to structural constraint.

TIM The fiber space in Figure 21 is embedded/associated with the discrete topological space in Figure 19 and act as a single algebraic object isomorphic to the structure of fiber bundle; as well as computationally expressed as TIM, as structured in Theorem 6. The Figure 22 further puts everything together in a broader perspective. The structure expresses all the class of behaviour of the Hardy model; possibilistic, non-possibilistic and $LC-GI$ which corresponds to feasible (black), infeasible (red) and contextual (blue) respectively. Since, Hardy Model is logically contextual so there are some path that are possible in the fiber space. Following this overview should facilitate the readers to understand other models as well, for example in Kochen-Specker model there are blue dotted lines, which gives one iteration as in the Hardy case. The whole discrete manifold of genus$=6$ represents the Kochen Specker as shown in Figure 28. The model is strongly contextual with no possible path in state space as shown in Figure 27, hence strongly contextual. We represent blue colour for $LC-GI$, black for feasible and red for infeasible computation

An equivalent way to understand this synergistic interaction between discrete structure and the computation is through the concept of symmetry. The irreducible representation of the symmetry group of a polyhedron representing all relevant combinations of the contexts of the possibilistic table of each empirical model is constrained by the critical and virtual simplices of the discrete structure. The structural constraint on symmetry decides whether there exists an intertwining transformation between its irreducible representations. The critical simplices represent forbidden transformations and emergence of virtual loop represent an *essential change* in the shape of the polyhedra itself which is a symmetry breaking process. We also provide a stratagem of our overview in Appendix C.

6.2 The Hardy Model

Our goal is to first construct a simplicial complex $\mathcal{K}$ from the given Hardy table in Figure 14 and then address the permutation in state space subject to $\mathcal{K}$ following the above stratagem based on main results of Section 3. We provide an extensive explanation of Hardy model and constructive proofs of other models will follow same method. Correspondingly, see construction of other models including Kochen Specker structure in subsection 28 and computation in subsection 27, Greenberger-Horne-Zeilinger structure in subsection 25 and computation in subsection 26 and Popescu-Rohrlich structure in subsection 23 and computation in subsection 24.

A,B	(00)	(01)	(10)	(11)
(a,b)	1	1	1	1
(a,b')	0	1	1	1
(a',b)	0	1	1	1
(a',b')	1	1	1	0

Figure 14: The Hardy table

The first phase of the stratagem to construct the structure component of TIM is the following:

Combinatorial description of the Hardy table A simplicial complex $\mathcal{K}_1$ is constructed from the measurement set $\{a, a^{'}, b, b^{'}\}$ by considering its power set which is equipped with a discrete topology as shown in Figure 15. $\mathcal{K}_1$ provides all possible ways in which the basic set-up of the table can be performed. One of the possible set-up would be when A selects $a^{'}$ and B selects b at time t. Afterwards at time $t^{'}$, A then selects a and B selects $b^{'}$, represented as $\mathrm{a}^{'}\mathrm{bab}^{'}$ as the first level of $\mathcal{K}_1$, similarly other three levels. $\mathcal{K}_1$ gives more information about different possible contexts when simultaneously measurement which is not determined directly in the table because $\mathcal{K}_1$ also considers three and four contexts, e.g., $\mathrm{ab}^{'}\mathrm{a}^{'}$. It say, A chooses a and B chooses $b^{'}$,while B is choosing $b^{'}$, A in the different possible set-up chooses $a^{'}$. It can also take into account when one of the agents doesn't measure in a given round. For example, consider Δ a$^{'}$bb of first level of $\mathcal{K}_1$, where A chooses $a^{'}$ and B chooses b, afterwards A doesn't choose anything while B chooses b. Pertinently, $\mathcal{K}_1$ takes into account all the possible ways (paths) in which the set-up of the table can be performed.

The contexts bb, b$^{'}$b$^{'}$, aa and a$^{'}$a$^{'}$ are not considered in the table therefore don't change the topological space associated with the table. We could use cell complexes (CW-complexes) that can avoid these contexts instead of $\mathcal{K}_1$ but both give same topological information. For example, the torus triangulated using $\mathcal{K}$ yields same number of critical simplices in each dimension using DMT as it would using cell complexes. Both have same topological description. Notice, we use *italics-type* a, b representing measurement set/contexts in contextual semantics. Its combinatorial counterpart is represented in normal-type a, b in the succeeding diagrams. $\mathcal{K}_1$ has a topology of a real line or disc because its Euler characteristics χ is 1, vertices $v = 16$, edges $e = 33$ and faces $f = 18$, so $\chi = v - e + f = 1$. It has trivial (co)homology and simplicial collapse to a vertex, so does not provide any significant class of behaviour of the table. The linear topology recalls tape structure of the Turing machine characterising local hidden variables which cannot reproduce contextuality due to Bell-Kochen-Specker theorem.

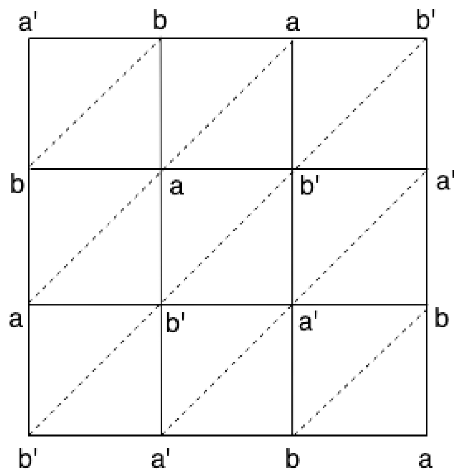

Figure 15: Combinatorial description of Hardy table: $\mathcal{K}_1$.

Axiom of choice and context indistinguishability Now, we would require to turn non-possibilistic events and $LC - GI$ of the table as critical simplex (red) and virtual simplex (blue) repectively over $\mathcal{K}_1$ in order to express each classes of behaviour. Suppose we randomly turn a particular context, say $ab^{'}$, of $\mathcal{K}_1$ critical (red) corresponding to the non-possibilistic event of the table. There arises two issues: context indistinguishability and axiom of choice of empirical models as discussed in the overview 6.1. The issue of context indistinguishability means when we randomly choose $ab^{'}$, there are six $ab^{'}$ indistinguishable contexts over $\mathcal{K}_1$. Moreover, choosing any of these contexts randomly would force fiber space to contain specific outputs constraining the axiom of choice, i.e., if one makes an edge as red between $ab^{'}$ over $\mathcal{K}_1$, it would mean that each corresponding fiber would consist only 0 (i.e., at (00)), because at (11) the given context is possible.

Let us say we randomly chose $ab^{'}$ of second level (row) of $\mathcal{K}_1$ as critical. Each row of $\mathcal{K}_1$ is a level in the combinatorial structure represented by a natural number, here we have four levels corresponding to four rows of $\mathcal{K}_1$. Following this notion, $ab^{'}_2$ would mean $ab^{'}$ of 2nd level. For $\mathcal{K}_1$ to compute (independently from fiber space consideration) which particular context $ab^{'}$ among six $ab^{'}$ is turned critical, we choose any vertex say a$^{'}$ and consider all loops, paths starting and ending at same vertex, and then map out which one particular context has not been considered in this path traversing set using set difference or complement of set operation, computing the particular critical context. The particular context here a$^{'}b$ is not given but computing this loop-set would automatically evaluate the particular context that was randomly chosen. The interest in loops is due to fundamental group being the semantic description of the structure quantifying homotopy information in terms of loops.

Below are all loops while choosing a$^{'}$ as starting vertex.

$L_1 = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_1\mathrm{b}_1^{'}\mathrm{a}_2^{'}\mathrm{b}_2^{'}\mathrm{a}_1\mathrm{b}_1\mathrm{a}_1^{'}$	$L_2 = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_1\mathrm{b}_1^{'}\mathrm{a}_2^{'}\mathrm{b}_2^{'}\mathrm{a}_1\mathrm{b}_1\ \mathrm{a}_2\mathrm{b}_2\mathrm{a}_1^{'}$	$L_3 = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_2\mathrm{b}_2\mathrm{a}_1^{'}$
$L_4 = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_2\mathrm{b}_3^{'}\mathrm{a}_3\mathrm{b}_2\mathrm{a}_1^{'}$	$L_5 = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_2\mathrm{b}_3^{'}\mathrm{a}_4^{'}\mathrm{b}_4^{'}\mathrm{a}_3\mathrm{b}_2\mathrm{a}_1^{'}$	$L_6 = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_1\mathrm{b}_1^{'}\mathrm{a}_2^{'}\mathrm{b}_3\mathrm{a}_3^{'}\mathrm{b}_3^{'}\mathrm{a}_3\mathrm{b}_2\mathrm{a}_1^{'}$
$L_7 = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_1\mathrm{b}_1^{'}\mathrm{a}_2^{'}\mathrm{b}_3\mathrm{a}_4^{'}\mathrm{b}_4\mathrm{a}_4^{'}\mathrm{b}_4^{'}\mathrm{a}_3\mathrm{b}_2\mathrm{a}_1^{'}$	$L_8 = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_1\mathrm{b}_2^{'}\mathrm{a}_3^{'}\mathrm{b}_4\mathrm{a}_4^{'}\mathrm{b}_4^{'}\mathrm{a}_3\mathrm{b}_2\mathrm{a}_1^{'}$	$L_9 = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_1\mathrm{b}_2^{'}\mathrm{a}_3^{'}\mathrm{b}_3^{'}\mathrm{a}_3\mathrm{b}_2\mathrm{a}_1^{'}$
$L_{10} = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_1\mathrm{b}_1^{'}\mathrm{a}_2^{'}\mathrm{b}_3\mathrm{a}_3\mathrm{b}_2^{'}\mathrm{a}_1\mathrm{b}_1\mathrm{a}_1^{'}$	$L_{11} = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_1\mathrm{b}_1^{'}\mathrm{a}_2^{'}\mathrm{b}_3\mathrm{a}_3\mathrm{b}_3^{'}\mathrm{a}_2\mathrm{b}_1\mathrm{a}_1^{'}$	$L_{12} = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_1\mathrm{b}_1^{'}\mathrm{a}_2^{'}\mathrm{b}_3\mathrm{a}_4^{'}\mathrm{b}_4\mathrm{a}_3^{'}\mathrm{b}_2^{'}\mathrm{a}_1\mathrm{b}_1\mathrm{a}_1^{'}$
$L_{13} = \mathrm{a}_1^{'}\mathrm{b}_1\mathrm{a}_1\mathrm{b}_1^{'}\mathrm{a}_2^{'}\mathrm{b}_3\mathrm{a}_4^{'}\mathrm{b}_4\mathrm{a}_4^{'}\mathrm{b}_3^{'}\mathrm{a}_2\mathrm{b}_1\mathrm{a}_1^{'}$		

Let L be set of all such paths $L = \sum_{n=1}^{13} L_n$. Any other path, if any, would be subset of the L-set. Let $\mathbf{L}$ be universal set of all paths of $\mathcal{K}_1$, then using complement of these sets or different between $\mathbf{L}$ and L gives set of contexts that are critical. It turns out naturally that context $ab_2^{'}$ is critical that was randomly chosen in $\mathcal{K}_1$ using loop information.

It would be obsolete to randomly turn any set of context critical based on the non-possibilistic events of the table. We would require a theory that would provide a systematic way to turn a set of particular simplices critical (red). As a result, the structure $\mathcal{K}_1$ would also put constraint on the maximum number of critical simplices to be included over $\mathcal{K}_1$. It means, the structure $\mathcal{K}_1$ would not allow random assignment of contexts to be turned red, rather would require union of discrete structures to be added or enumerated to contain and express all class of behaviour of the table based on the conditions imposed on $\mathcal{K}_1$. Computationally, we would iteration $\mathcal{K}_1$ into different intermediate $\mathcal{K}$'s to include all class of behaviour of the table, quantifying the evolution description of $\mathcal{K}_1$. We use discrete Morse theory to put constraint on the structure $\mathcal{K}_1$.

$\mathcal{K}_1$ is homeomorphic to the real line topology or disc which would make one context critical, say $ab^{'}$, precisely $ab_2^{'}$, i.e., $ab^{'}$ of 2nd level based on DMT. Now, we exactly know which of the particular context among other similar indistinguishable contexts have been turned critical via discrete Morse function. Technically, computing the fundamental group of $\mathcal{K}_1$ (homeomorphic to a disc) which is the semantic description of the structure in TIM, turns out to be identity. It would allow direct tensor product of different fiber bundles which makes this category into a symmetric monoidal category, in fact a distributed monoidal category. The category does not take into account the topological information as is the case of braided monoidal category which is the categoric description of TIM. Moreover, the essence of fundamental group is quantified as loops over $\mathcal{K}_1$ but using DMT over $\mathcal{K}_1$ does not allow loops. One of the natural way is identification of the boundary of $\mathcal{K}_1$ which in turn topologises $\mathcal{K}_1$ from a trivial disc or real line topology to a surface, turning $\mathcal{K}_1$ to $\mathcal{K}_2$ as in the Figure 16. Along side to each state space associated with each vertex, one associates all possible values of outputs to give axiom of choice which is central to contextuality, i.e., each state space (fiber) contains both 0 and 1 allowing all possibilities between any contexts.

$\mathcal{K}_2$ is homeomorphic to a torus which is a reasonable minimal surface for expressing contextuality. Infact, the canonical commutation relation between two observables has a toric representation. Noncommutative torus serves as a fundamental object for noncommutative space and C* algebra. Torus encodes naturally the obstruction (holes) led sheafication of contextuality unlike positively curved spaces like sphere.

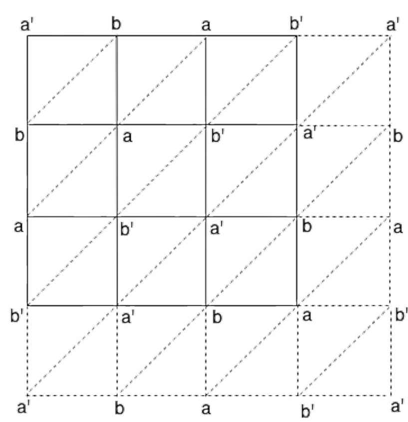

Figure 16: Topologising $\mathcal{K}_1$ to $\mathcal{K}_2$.

Evolution of the structure The goal is to enumerate $\mathcal{K}_2$ until the structure includes the class of behaviour expressed by the Hardy table. In TIM computational description, one can consider $\mathcal{K}_2$ as starting base space over which computation in the fiber space would take place. Not all non-possibilistic events from Hardy table could be included over $\mathcal{K}_2$ due to constraint from DMT. $\mathcal{K}_2$ would evolve into different intermediate $\mathcal{K}$'s until the structure includes every non-possibilistic and $LC - GI$ event as critical simplices and virtual loops respectively based on computation in the state space.

If one calculates its Euler characteristics χ of $\mathcal{K}_2$, the number of vertices v are 25, edges e are 89 and faces f are 64, $\chi(\mathcal{K}_2) = v - e + f = 25 - 89 + 64 = 0$, which is Euler characteristic of torus. The boundary of the simplicial complex is identified. It is homeomorphic to a torus. Based on the construction of DMT, the contexts $ab^{'}$ and $a^{'}b^{'}$ are non-possibilistic shown in red as critical simplices. There is also one 2-simplex shown as critical based on three contexts at a time which is not expressed in the table as shown in Figure 17. Any of the critical simplices will only be considered effective based on the allowed permutation over fiber space discussed in second phase of the generalisation.

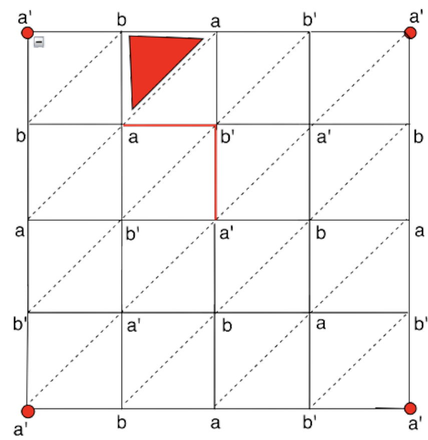

Figure 17: DMT on first enumeration of $\mathcal{K}_2$

The space constrains the way contexts are connected so there remains another non-possibilistic event $a^{'}b$ which cannot be expressed in 1-torus due to constraints put by $\mathcal{K}_2$. As a result, we enumerate $\mathcal{K}_2$ to $\mathcal{K}_3$ so as to consider context $a^{'}b$, which is a 2-torus as shown in Figure 18. It has $v = 32$, $e = 106$ and $f = 72$ with $\chi(\mathcal{K}_3) = -2$. Notice, the space evolves from 1-torus to 2-torus and not to a disc, it is due to aforementioned reasons.

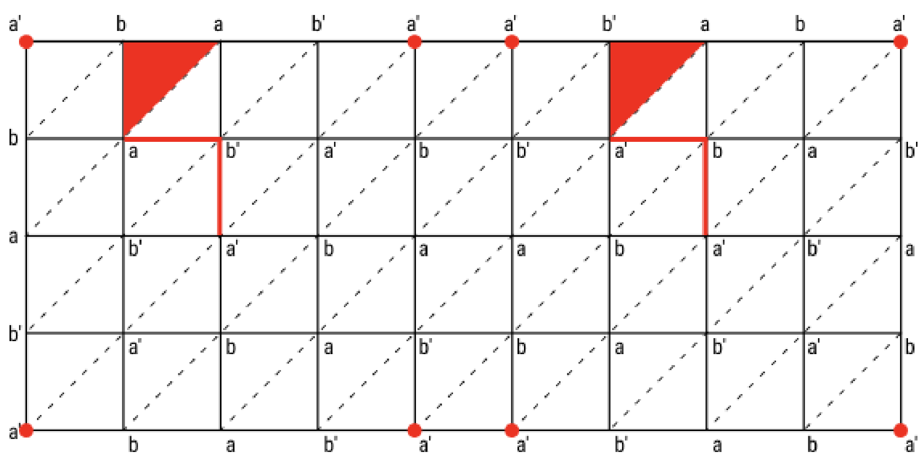

Figure 18: Second enumeration for containing other non-possibilistic contexts of Hardy table: $\mathcal{K}_3$.

The 2-torus doesn't take into account the context (a, b) which is LC-GI in the table which will be expressed as a virtual short-lived loop. LC-GI would mean an edge between (ab) at (00) is possible because there is no critical simplices but due to virtual loop it is not possible.

The possibility of non-trivial virtual loops emerge due to equidistant critical points. The edge joining points $a^{'}$ and $a^{'}$ marked as blue has a choice to go in either directions. Let's say, at some time it chooses different direction (marked by red arrow). It turns out that there is a possibility of a non-trivial virtual loop for example triangle *bba* taking into account the context (a, b). Following the construction, $\mathcal{K}_3$ is enumerated to $\mathcal{K}_4$ which is homeomorphic to a 3-torus. $v = 48$, $e = 164$ and $f = 112$ with $\chi(\mathcal{K}_4) = -4$. The resulting simplicial complex $\mathcal{K}_4$ based on contexts of the table effectively triangulates a topological space X homeomorphic to a 3-torus as in Figure 19.

Notice, there are two classes of virtual loops: one class at the boundary of topological transition of $\mathcal{K}_2$ realised as 1-torus to $\mathcal{K}_3$ realised as 2-torus, and other class at the boundary of topological transition of $\mathcal{K}_3$ to $\mathcal{K}_4$ realised as 3-torus. These topological transition characterised through virtual loops changes the homotopy class of the structure. The virtual loops do not violate causality and the definition of discrete Morse function. The change of homotopy class of space could be structured conceptually as a symmetry breaking process.

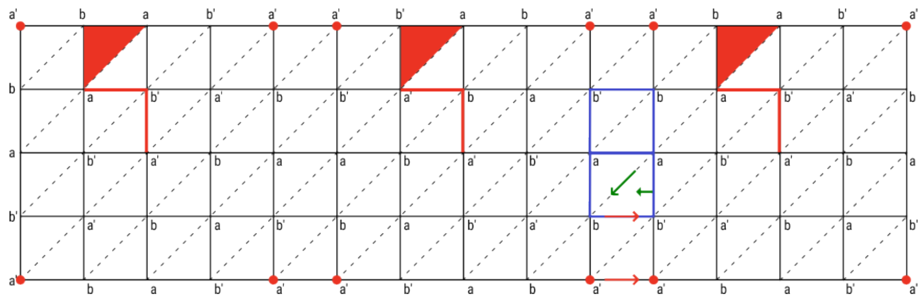

Figure 19: Final enumeration: Topological environment of Hardy Model. The red simplices represent infeasible computation, blue region represent possible non-trivial loops for contextual behaviour and black simplices show feasible computation: $\mathcal{K}_4$.

The final enumeration as $\mathcal{K}_4$ in Figure 19 expresses all possible behaviour of the table. One can further refine $\mathcal{K}_4$ to express same class of behaviour on a minimal surface to avoid redundant critical simplices represented as $\mathcal{K}_5$ as shown in Figure 20 homeomorphic to a 2-torus. The relevance of every simplices would further depend on the contexts in the state space. $\mathcal{K}_5$ cannot be directly constructed iteratively from the table but after one constructs $\mathcal{K}_4$ from the above framework. For instance in the first iteration of $\mathcal{K}_5$ of Figure 20, we have two $a^{'}$b which would be inconceivable a priori without resorting to $\mathcal{K}_4$ first. So, $\mathcal{K}_5$ characterises the Hardy model but since $\mathcal{K}_4$ is a priori requirement to turn it into $\mathcal{K}_5$ we put $\mathcal{K}_4$ for this reason in Figure 22.

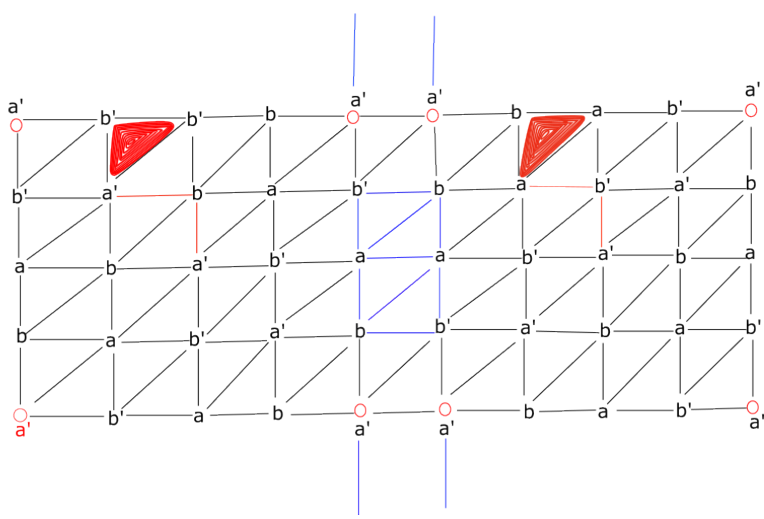

Figure 20: Possible refinements in $\mathcal{K}_4$ turning to $\mathcal{K}_5$

Each vertex of the $\mathcal{K}_5$ realised as 2-torus has a fiber attached to it. It consists of all possible outputs providing element of choice for the agents. Broadly, the space consists of three classes of behaviour, one class is the subset of generic simplices which represents possibilistic/feasible behaviour, the second class is subset of critical simplices which represents the non-possibilistic/infeasible behaviour and third class is set of non-trivial paths which represents LC- GI behaviour. Instead of accounting all the fibers over each vertex, we are interested to extract an irreducible behaviour that can be expressed over the space.

The second phase addresses the permutation of the state space and consists of the following steps:

State space for computation The state (fiber) space consists of all possible outputs considering the axiom of choice, which in case of the Hardy model are 0 and 1. The state space will contain two states in every fiber which would correspond to four possible permutation between every two vertices of the 2-torus. The semantics for the state space is the symmetry group which describes the permutation of possible outputs. Technically, considering every permutation between every vertices of the 2-torus is computationally hard. We are not required to consider every permutations because we are interested in the classification of permutations unfolding into an irreducible class of behaviour. Moreover, 2-torus represents the final space that evolved from the interaction between computation in state space. The core of this interaction started from the measurement set whose power set represented different possible combinations of the set-up. As a result, our focus is the the measurement set whose permutations correspond to the permutations in the state space iterating the space into 2-torus in the light of TIM framework. Recall, the permutation in fiber space is constrained by its corresponding simplex representing the context. In order to formalise the concept mathematically, the cardinality of the measurement set corresponds to the vertices represented structurally as a polyhedron. The cardinality of the measurement set in the Hardy model is four which would correspond to the regular tetrahedron.

We start with the measurement set $\{a, a^{'}, b, b^{'}\}$ consists of four elements which is represented as a regular tetrahedron. Each symmetry of the tetrahedron corresponds to a permutation of its four vertices which is represented as tetrahedral group. Every element of the tetrahedral group permutes the vertices of the regular tetrahedron among themselves. It accounts all the 24 symmetries of the tetrahedron along different planes. The group of all self-isometries that sends a regular tetrahedron to itself is isomorphic to S_4 symmetry group. So the group of all these symmetries corresponds to subgroups of S_4 and their inclusion in Galois sense subject to the structural constraint of 2-torus. The representation of S_4 represents its 24 symmetries as invertible matrices forming a group. Each element of the group is given its corresponding matrix which is represented over the state space (fiber) as a representation of group F. The arbitrary groups F_i where $i = 6$ are subgroups of S_4. The Theorem 13 is synthesis of this step in which transformation $\mathbf{T}$ is allowed based on the Theorem 4 expressing computation subject to the Theorem 2 expressing global constrain. The symmetries of tetrahedron have additional structural constraint from the structure of 2-torus as shown in Figure 21.

The linearised tetrahedron as in the right side of Figure 21 is the pictorial way of understanding the transformations; the fibers attached to each vertex is the state space which contains two outputs 0 and 1 and the base gives the relevant irreducible contexts. Mathematically, the fibers are the vector spaces associated to each vertex of the tetrahedron and the transformations in fiber space are matrix operation corresponding to the edge of the tetrahedron via representation theory. The fiber space includes all 24 possible transformations of the tetrahedron. The contexts are constrained by the structure of 2-torus. Any transformation in the fiber space is only possible if its corresponding context is not critical simplex or virtual loop in the structure of 2-torus.

Notice, It seems in Figure 21 that there are 28 possible transformation contrary to 24 permutations of tetrahedron. The reason is that the state space contains a repeated context of $ab^{'}$ which should be neglected and being a repeated context the nature of transformations will be similar to its original $ab^{'}$. In order to linearise all the edges of the tetrahedron one would go through one of its edges twice making one of its edge to be repeated. So, the 4 possibile permutations associated with this repeated context is not considered and hence yields relevant 24 permutations of the tetrahedron. The expressivity of the model does not change if one considers the repeated context independently of its original context as we preferred in Figure 21.

The computational state space of the Hardy table is isomorphic to S_4 symmetry group. The transformations are constrained by the structure of 2-torus. One can view the Hardy table weakly isomorphic to the character table of S_4. The character table of S_4 is locally consistent (LC) satisfying Schur orthogonality relations but there are inequivalent irreducible representations responsible for global inconsistency (GI).

Remark. *The local section correspond to any possible permutation of the tetrahedron which corresponds to any possible transformation in the state space of the linearised tetrahedron on right side of Figure 21 and the global section correspond to the transformations or symmetry operations that permutes the vertices to itself which corresponds to loops starting and ending at the same vertex represented as purple lines in Figure 21 similar to sheaf modelling framework.*

Since the fiber space is attached with the base space as shown in 21 is embedded in a discrete space of Figure 19. So, the transformation **T** over the fiber space depends on the type of simplices of its underlying space.

TIM framework The transformation in fiber space is based on its corresponding path over the discrete manifold. The critical simplices set is $\{aa^{'}b^{'}, a^{'}b, a^{'}b, aab, ab^{'}\ a^{'}b^{'}\}$ and the virtual loops set $\{b^{'}ba, aab, aab, abb^{'}\}$. The critical simplices are turned red in the fiber space and the virtual loops are turned blue in the state space of Figure 21. Any possible section is represented in black color and is the local section. The global section corresponds to operations permuting tetrahedron to itself which corresponds to loops represented in purple lines. Among these purple lines there are families of sections that are possibilistic or feasible because Hardy Model is logically contextual. Any critical simplex or any virtual loop which do not correspond to the irreducible relevant contexts of the linearised tetrahedron are not considered.

Remark. *The cardinality of measurement set of the PR boxes is also four and hence described by a regular tetrahedron but unlike the Hardy model has a topological realisation of* 4*-torus. There exists no family of global sections in the PR Boxes because the model is strongly contextual.*

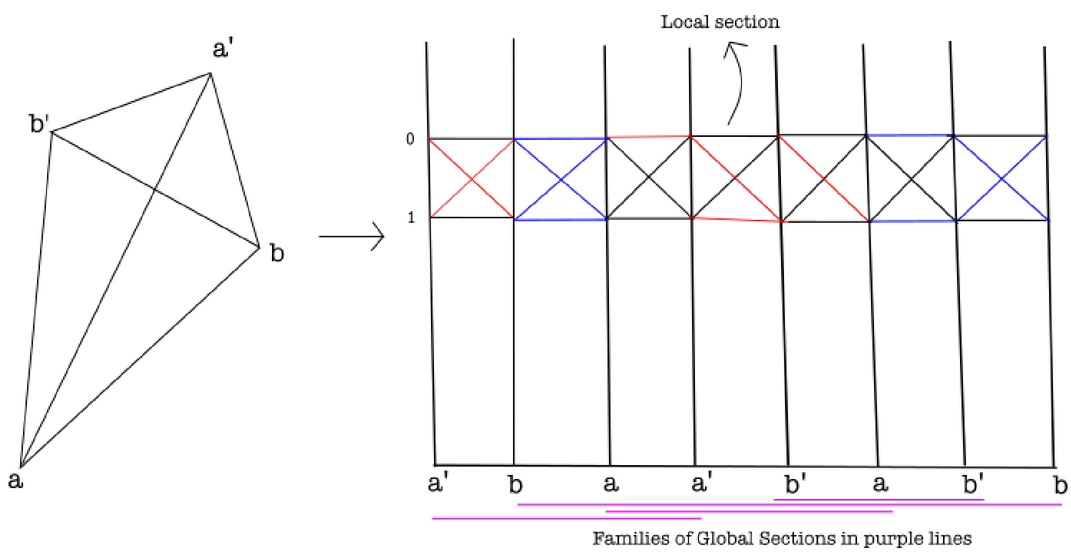

Figure 21: Hardy Model s logically contextual due to possible family of global sections in the state space. The set of critical simplices is $\{aa^{'}b^{'}, a^{'}b, a^{'}b, aab, ab^{'}, a^{'}b^{'}\}$ and set of possible virtual loops is $\{b^{'}ba, aab, aab, abb^{'}\}$

Theorem 4. *Symmetry group* S_4 *represents Hardy model computation.*

Proof. Proof by construction. □

Theorem 5. *Any empirical model representable by Hardy model type scenario has a topological representation of a surface of genus 2 and is logically contextual in nature.*

Theorem 6. *For the Hardy model* $\mathcal{G} = S_4 \rtimes \pi(T^3)$

6.3 Kochen Specker Model

Theorem 7. *Dihedral group* D_5 *represents computation of the Kochen-Specker model which represents the symmetry of the pentagonal bipyramid.*

Theorem 8. *Any empirical model representable by the Kochen-Specker type scenario has a topological representation of a surface of genus 6 and is strongly contextual in nature.*

Theorem 9. *For the Kochen-Specker model* $\mathcal{G} = D_5 \rtimes \pi(T^6)$

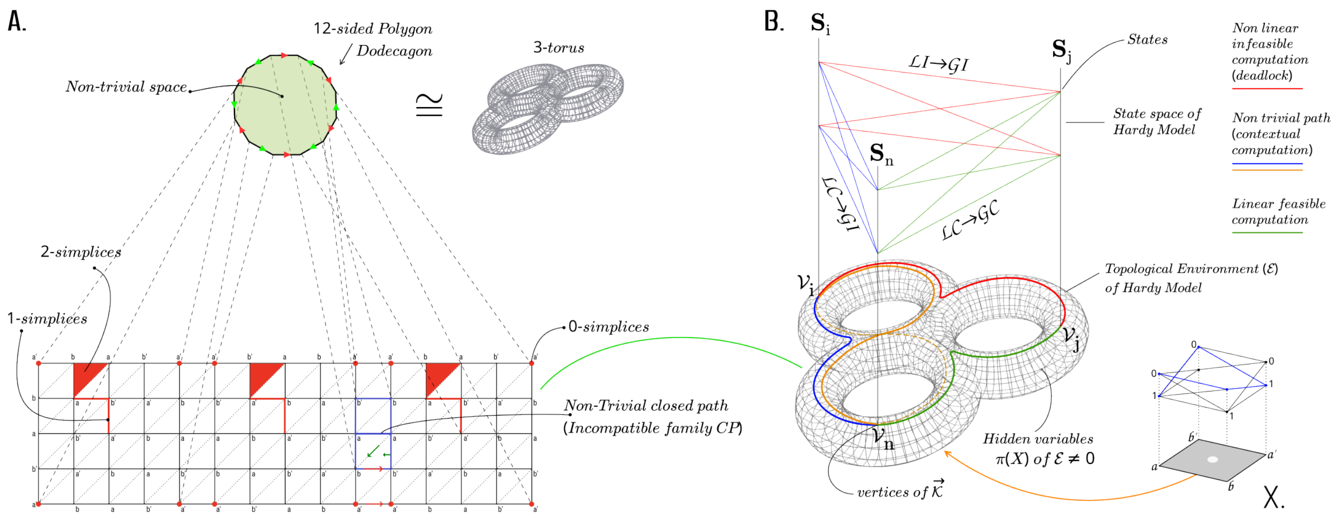

Figure 22: Topological realisation of the Hardy model HM as 3-torus. Here, we consider its description as 3-torus not its refined 2-torus because 3-torus is the natural space constructed via iterations in TIM framework. The right side represents the computation over hardy model with $g = 3$. The blue line $\mathcal{V}_1 \rightarrow \mathcal{V}_2$ has obstructions (two holes) that correspond to infeasible computation like a deadlock ($LI \rightarrow GI$), the green line $\mathcal{V}_2 \rightarrow \mathcal{V}_3$ has no non-linearity in between that corresponds to feasible computation ($LC \rightarrow GC$) and the red line $\mathcal{V}_3 \rightarrow \mathcal{V}_1$ goes around obstruction (holes) forming a non-trivial closed path as brown line which corresponds to locally consistent and globally inconsistent computation ($LC \rightarrow GI$). The brown arrow from X to B infers its base space as 3-torus constrains its computation.

6.4 Greenberger-Horne-Zeilinger Model

Theorem 10. *Octahedral group* O_h *represents computation of the Greenberger-Horne-Zeilinger model which represents symmetry of the octahedron.*

Theorem 11. *Any empirical model representable by the Greenberger-Horne-Zeilinger type scenario has a topological representation of a surface of genus 8 and is strongly contextual in nature.*

Theorem 12. *For the Greenberger-Horne-Zeilinger model* $\mathcal{G} = \mathrm{O}_h \rtimes \pi(T^8)$

6.5 Popescu-Rohrlich Boxes

Theorem 13. *Symmetry group* S_4 *represents computation of the Popescu-Rohrlich boxes which represents the symmetry of tetrahedron.*

Theorem 14. *Any empirical model representable by the Popescu-Rohrlich boxes type scenario has a topological representation of a surface of genus* 4 *and is strongly contextual in nature.*

Theorem 15. *For Popescu-Rohrlich boxes model* $\mathcal{G} = \mathrm{S}_4 \rtimes \pi(T^4)$

6.6 Peres-Mermin Magic Square

Theorem 16. *Octahedral group* O_h *represents computation of the Peres-Mermin Magic Square.*

Theorem 17. *Any empirical model representable by the Peres-Mermin magic square has a topological realisation of a 1-torus* T^1*, and is contextual in nature [23].*

Theorem 18. *For the Mermin square* $\mathcal{G} = \mathrm{O}_h \rtimes \pi(T^1)$

6.7 Proofs by Construction

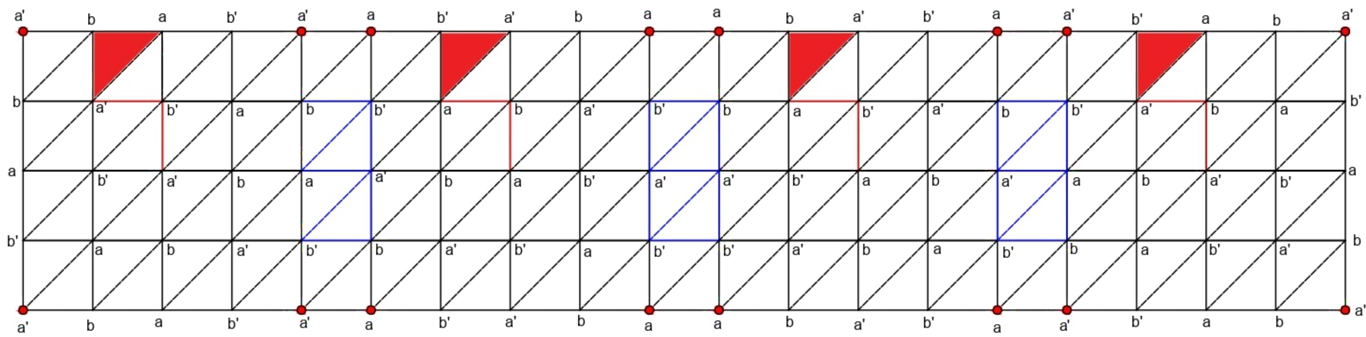

Figure 23: Popescu-Rohrlich box is represented by a surface with genus 4. The red simplices represent infeasible computation, blue region represent possible non-trivial loops for contextual behaviour and black simplices show feasible computation.

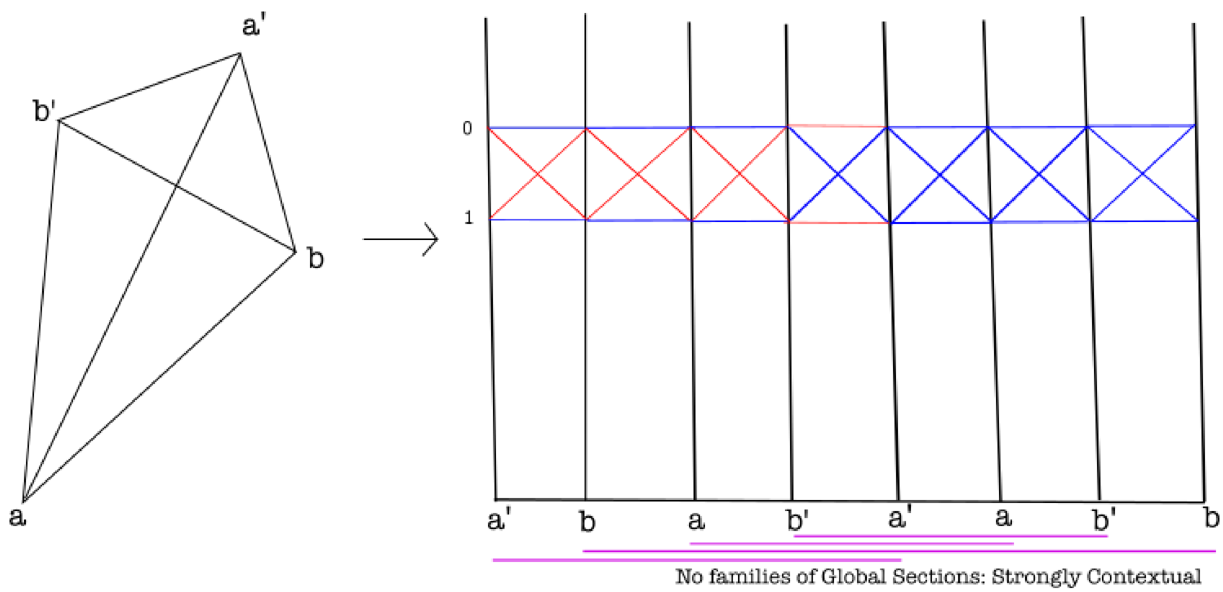

Figure 24: Popescu-Rohrlich computation. It is strongly contextual due to no possible global paths/computation over the state space. Red paths and blue paths represent infeasible and contextual computation respectively which corresponds to the red and blue simplices of its underlying discrete surface as in Figure 23. The state space of hardy model is also represented by a tetrahedron but unlike Popescu-Rohrlich box it allows computation in its state space due to discrete surface of genus 3 of Figure 21. The set of critical simplices is $\{baa^{'}\ a^{'}b^{'}, a^{'}b^{'}, b^{'}a^{'}a, ab, ab, ba^{'}a, ab^{'}, ab^{'}, b^{'}aa^{'}, a^{'}b, a^{'}b\}$ and set of virtual loops $\{bb^{'}a, aa^{'}b^{'}, aa^{'}b^{'}, a^{'}b^{'}b, b^{'}ba^{'}, ba^{'}a^{'}, a^{'}a^{'}b^{'}, a^{'}b^{'}b^{'}, bb^{'}a^{'}, b^{'}a^{'}a, a^{'}ab^{'}, abb^{'}\}$

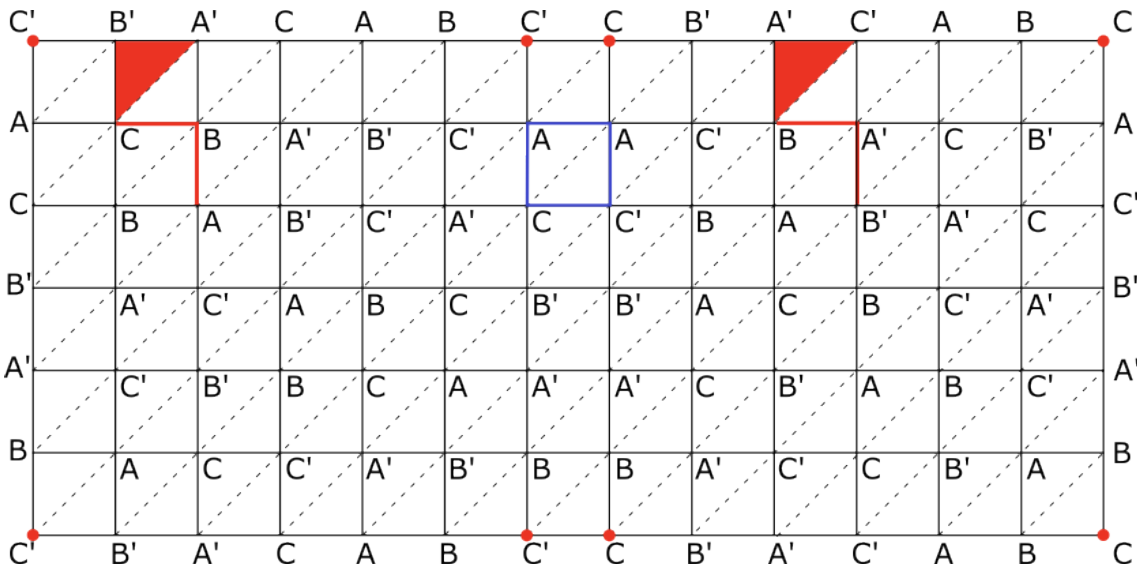

Figure 25: Greenberger-Horne-Zeilinger Model is represented by the surface of genus 8. The discrete surface is 4 times of the above figure but for simplicity, it is shown in a reduced way.

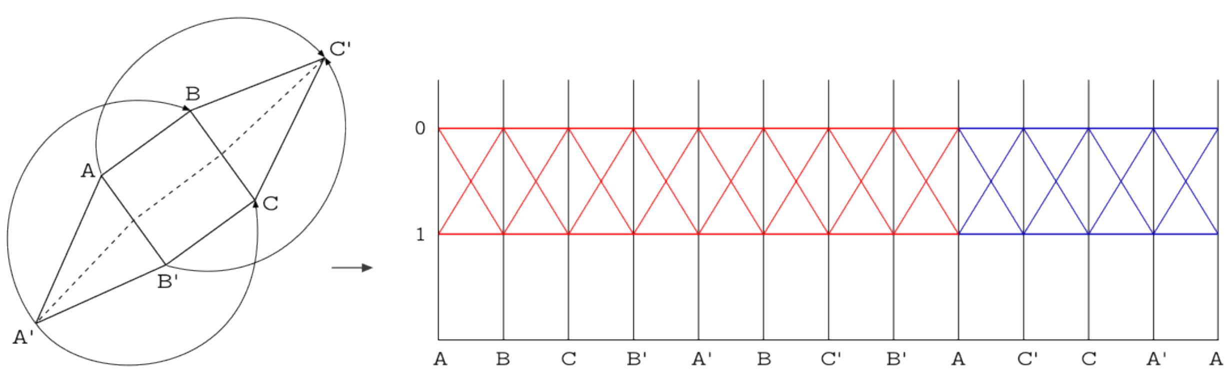

Figure 26: Greenberger-Horne-Zeilinger is strongly contextual with no possible global sections in the state space.The set of critical simplices is {B′A′C, B′A′C, B′A′C, B′A′C, CBA, CBA, CBA, CBA, A′C′B, A′C′B, A′C′B, A′C′B, BA′B′, BA′B′, BA′B′, BA′B′} and the possible set of virtual loops is { C′CA, C′CA, C′CA, C′CA, CAA, CAA, CAA, CAA, AAC, AAC, AAC, AAC, ACC′, ACC′, ACC′, ACC′, CC′B′, CC′B′, CC′B′, CC′B′, CB′B′, CB′B′, CB′B′, CB′B′, B′B′A′, B′B′A′, B′B′A′, B′B′A′, A′A′B′, A′A′B′, A′A′B′, A′A′B′, A′A′B, A′A′B, A′A′B, A′A′B, A′BB, A′BB, A′BB, A′BB, BBC′, BBC′, BBC′, BBC′, BC′C, BC′C, BC′C, BC′C} based on its associated discrete space as in Figure 25.

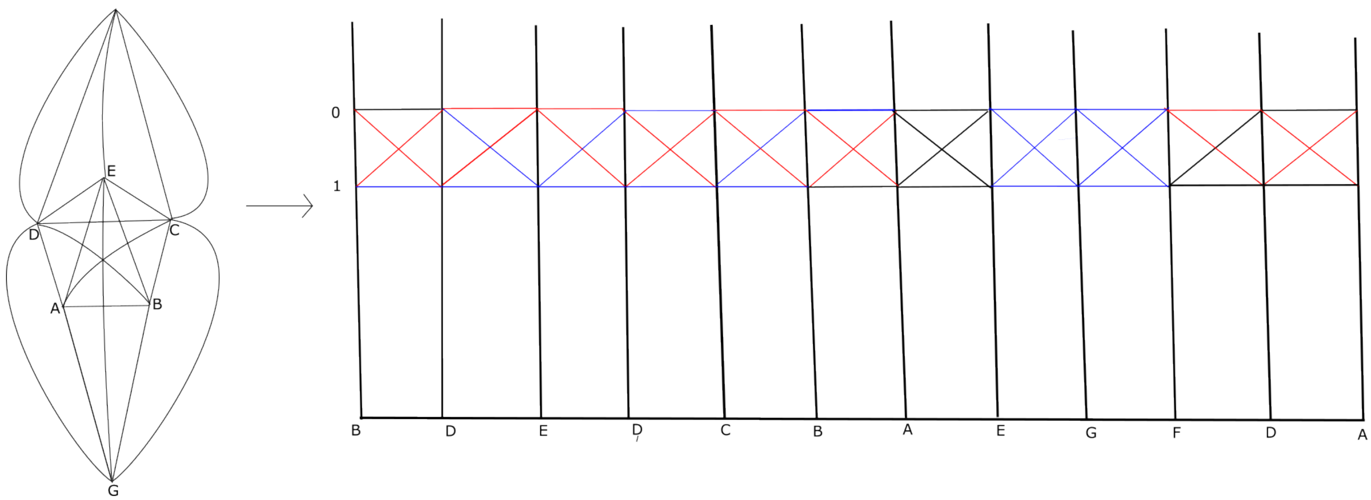

Figure 27: Kochen Specker model is strongly contextual with no possible global section in the state space. The set of critical simplices and virtual loops is given from the structure shown in its associated discrete manifold of Figure /refks1. The set of critical section is {ACB, BDE, ACB, BDE, CED, CED, DBF, DBF, CEA, CEA,ADF, ADF,} and possible set of the virtual loop is {FAG, AAG, AAB, ABB, BBC, BCC, DDE, CCD, CDD, DEF, EFG, GEF, GEF, EFG, GFA, AAF, AAB, BBA, BBC, CCB, CCD, DDC, DDF, DFG, GFE, GEE, EEG, GFE, FFA, AAF, AAB,BBA, BBC, BCC, CCD, CDD, DDG, DGG, GGE, GEE, EEF, EFF, FFA, ABF, ABB, BBA, BAC, ACC, CCD, CDD, DDG, DGG, GGE, EGE, EEF, FFE, FFB, BBF, BBA, BAA, AAC, CCA, CCD, DDC, DDG, DGG, GGE, EEG, EEF, FFE }. Note, that we do not consider all of the possible permutations but rather only those contexts that are significant and considered in the Kochen-Specker table. We have provided all possible permutations of the contexts in Appendix D. It turns that there is no possible family of global sections given all possible permutations constrained to the surface of Figure 28.

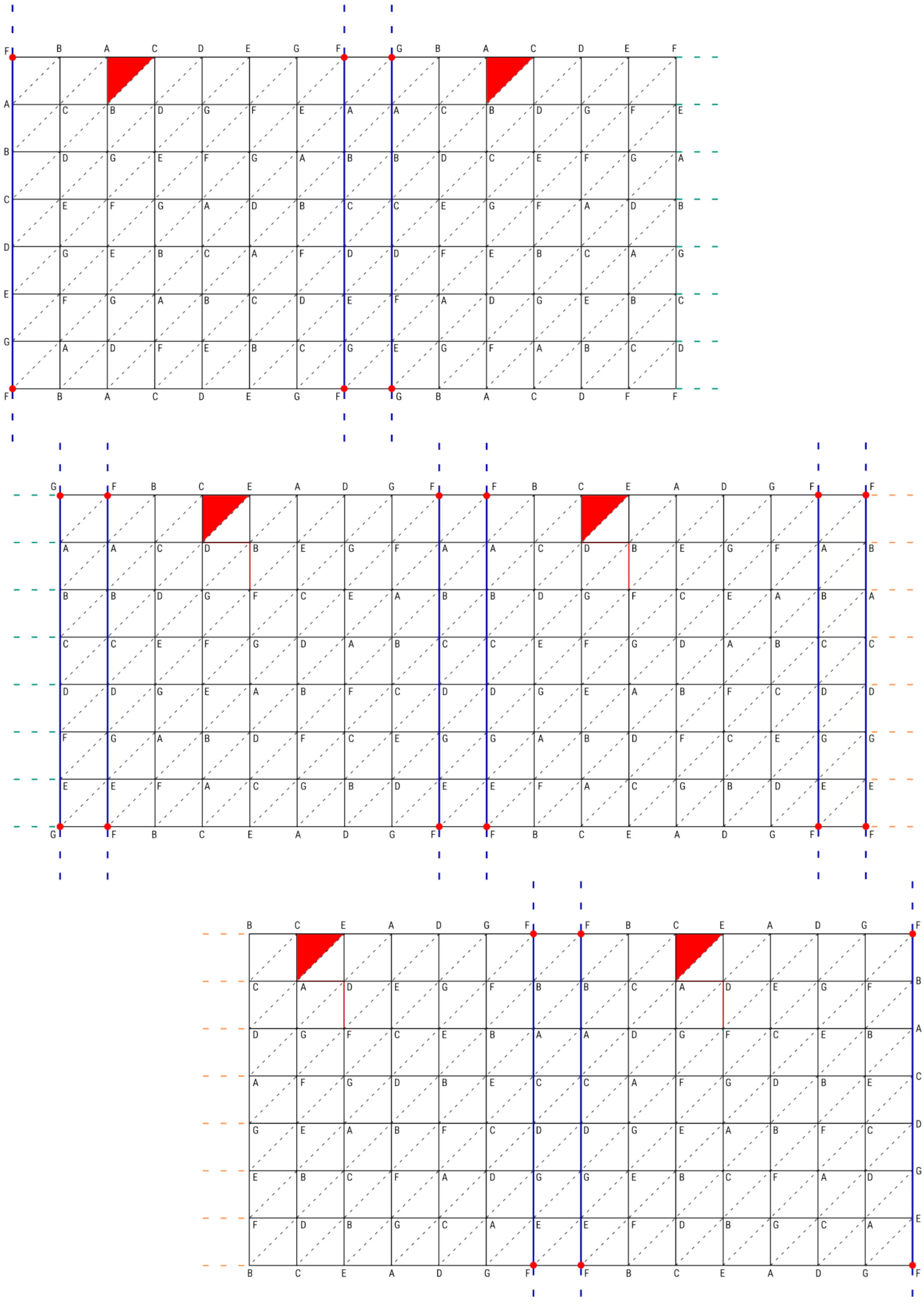

Figure 28: The whole discrete manifold represents the Kochen Specker model having genus= 6. The model is strongly contextual with no possible global section in state space as shown in Figure 27.

7 Discussion

These results could be explored in several promising direction.

1. *Quantum Advantage:* The results of the study can provide a way to explore the foundations of quantum advantage. The virtual loops in the discrete manifold are responsible for $LC - GI$ are sufficient to construct the ambient topological space associated with models. It means information about these class of loops could exempt most of the redundant information leading to quantum advantage in the computationally hard tasks.
2. *Computational Complexity:* It would be interesting to measure the computational complexity of the TIM in order to quantify quantum advantage as resource. In particular, we would like to compute the Jones polynomial in the context of the TIM which is a braided monoidal category, whose background mathematical structure is operated by a topological field theory, emerging as a potential candidate to compute these polynomials. The machine can use loops as a resource for quantum advantage which could possibly give a near polynomial-like time algorithm to compute the Jones polynomial.
3. *Formalising Openness:* These preliminary results advocates the concept of openness that emerges naturally in TIM; but we did not generalise it. A generalisation would require to formalise the dynamical aspects of synergistic computation-structure interaction which is still open to resolve.
4. *Computational Invariant:* These virtual loops are not explained and characterised in a mathematical and physical. It could prove to be a computational invariant which could be realised as a new operation in structural process algebra theory proposing an essential structural change in concurrency theory.
5. *Computational Quantum Theory:* It would be interesting to explore questions like, *can we interpret the significant aspects of quantum mechanics over TIM?* The expressiveness of contextual semantics machinery could enable a deeper look into the foundations of computability and serve as a step to explore new computational interpretations of quantum physics.
6. *Implementation:* Topological insulators could be a close prototype to understand the scope of TIM implementation. The virtual loops invokes boundary dynamics which in the light of these insulators could find a potential step to use exotic states of matter in a highly correlated system to engineer and control for ultrafast, noise resistant and cost effective computing over quantum computer. Also, the topological defects in condensed matter physics could further allow deeper probes to circuit designing for TIM. The connection can be explored in the light of topological circuits and non-commutative linear logic which could provide implementation and programming principles for hypercomputing.
7. *Inverse Problem:* Given an empirical model, we constructed a discrete manifold which is part of the computational model; what about reverse?, i.e., given a discrete manifold and a character table of a symmetry group encoding computation over the structure, can we construct new table of correlations? It could allow predicting new possible class of empirical models that could be in future discovered by some physical experiment.
8. *Automata theory:* One could seek connection of contextual semantics with undecidability in terms of Rice's theorem. Also, exploring a formal answer to the question posed by field medalist Terence Tao: can a group be a universal Turing machine. It would be interesting to quantify our work at the interface of automata and group theory.
9. *Biological modelling:* We expect our work to be applied in modelling virtual cell and neuronal dynamics for extracting hidden patterns. The scope of work could be also seen in theoretical and mathematical biology and to community of reaction systems to reason about concepts like intentionality and perception in nature-inspired model of computation.

Related work

The notion of context is pervasive in both computation and quantum physics. The concept can be seen in the models of computation and quantum contextuality respectively. *Computable topology* is an exclusive broad community that studies the topological and algebraic structure of computation. The topological aspect of quantum contextuality on the other side has also been extensively developed [5, 23]. The generalisation of quantum contextuality broadens the scope to understand *structure beneath the collective context* which emerges topological models of computation [15]. The relevant applications of contextual semantics in computer science include relational database [3], natural language semantics [9], robust constraint satisfaction [7] and logic [2]. Recently, a basic qualitative element of contextuality has been incorporated in the PTM [20]. The formal connection between interactive computation and quantum physics is explored on the basis of *interaction as observation* which could enable deeper look in the foundations of computability. The interested readers may also refer to the joint work of Merelli and Rasetti for background of our approach [25].

Author Contributions: The first author formalised mathematical connection between quantum contextuality and interactive computation and the second author described the concept of interactive model in a wider scope of topology.

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Conflicts of Interest: The author declares no competing interest.

A Appendices

A.1 Fiber bundle Example

Let's take a simple example of a cylinder in order to understand fiber bundles. We choose one particular circle S_1 denoted in pink in the middle of this hollow cylinder as shown in Figure 29. With S_1 in place, every point on the cylinder, like point p, can be given a coordinate system, by associating a point on S_1 and height above it. If we take height at S_1 as zero, the height above S_1 is positive and below is negative. So, p has two coordinates (a, b), $a \in S_1$ and $b \in \mathbb{R}$; this ordered pair on the surface of the cylinder identifies every point on it. The cylinder can be broken down into three separate spaces ; the whole cylinder as the total space E, S_1 as the base space and $\mathbb{R}$ as fibers. The idea is that the total space can be constructed from the base space and the fiber space; i,e., $E = S_1 \times \mathbb{R}$.

Notice that every point on the cylinder is an ordered pair of S_1 and $\mathbb{R}$. A particular fiber $\mathbb{R}_a$ associated with p is really not the same as other fiber $\mathbb{R}_b$ associated with other arbitrary point, say q; but both are isomorphic to generic $\mathbb{R}$. So we talk about fiber in two ways; as a fiber of a point or the standard fiber for which all the fibers are identical copies, i.e., diffeomorphic. Each fiber is associated with each point in the total space via projection map π, i.e., $\pi : E \to B$ with $\pi(a, b) = a$ and $\pi^{-1}(a) = \mathbb{R}_a$. Take another point q, with coordinates (c, d), so, $\pi(c, d) = c$ and $\pi^{-1}(c) = \mathbb{R}_c$. Both $\mathbb{R}_a$ and $\mathbb{R}_c$ are different sets, at the same time diffeomorphic to the generic fiber $\mathbb{R}$ – the local trivialisation condition, i.e., there should always be a way to be able to divide the total space locally, i.e., the image of E has to be diffeomorphic to $S_i \times \mathbb{R}$.

The construction of the total space of a cylinder is the cartesian product of the base space and the fibers, i.e., it is has a product topology because is a trivial space; locally a product space and globally a product space. Imagine the imaginary lines perpendicular to S_1 has a infinitesimal thickness of $a \in S_1$ and length of $\mathbb{R}_a$, i.e., $S_1 \times \mathbb{R}_a$ locally; and globally gluing together these infinite strips, unions and intersections yield the global topology

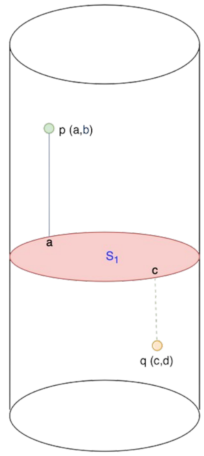

Figure 29: An Example

from global cartesian product. This is *local consistency and global consistency* because of linear global structure; but generally the total space could be complex.

For a general situation, any given subset $\mathcal{U}_i$ of B that is part of its cover, there is a pre-image in E, as shown in the Figure 30. The image of E is diffeomorphic to $\mathcal{U}_i \times F$ due to local trivialisation condition. We skip the technical details here for the sake of brevity. Any local (open) region of E is diffeomorphic to product of open sets in B and fiber F via π and ψ respectively. There will be other open set, say $\mathcal{U}_j$ in B with its pre-image in E and corresponding diffeormorphism but under different mapping, here σ. The two different diffeomorphisms, i.e., $\mathcal{U}_i \times F$ and $\mathcal{U}_j \times F$ don't have to necessarily agree over the overlap at $\mathcal{U}_i \cap \mathcal{U}_j$, as shown in Figure 30, as a purple dotted red line. A point in the intersection under one map could land in a completely different place in other map. We define a map from $F \to F$ that converts the two different local trivialisations into one another; resolving intersections; a map from F to itself, converting F that is generated via ψ mapping to be able to translate to σ mapping, i.e., $\psi \cdot \sigma^{-1}$; formally known as structure group of the fiber bundle and for cylinder it is identity. So, generally there is always a local trivialisation but globally no action between two fibers because of non-linearity in B which is responsible for *local consistency and global inconsistency.* It is always possible to describe any region of the E as the cartesian product of B and the standard fiber F region; but the mapping between the different cartesian products with mutual non-empty intersection is not always possible. Local consistency condition in fiber bundle is the local trivialisation condition; $\psi : \pi^{-1}(\mathcal{U}_i) \to F \times \mathcal{U}_i$ condition and $\psi_j^{-1} \cdot \psi_i : F \times (\mathcal{U}_i \cap \mathcal{U}_j) \to F \times (\mathcal{U}_i \cap \mathcal{U}_j)$ is the global consistency condition, whose transitions belong to the structure group $\mathcal{G}$ of the fiber bundles. The gauge group $\mathcal{G}$ measures non-locality and contextuality from non-trivial elements of the group.

Transition from local to global section: The fiber bundle is locally a product space but globally can have a complex topology; which provides a passage from local to global in the same sense as in sheaf theory; A section κ of a fiber bundle gives an element of the fiber over every point of X. Its a map $\kappa : X \to E$ such that $\pi \cdot \kappa$ is identity in X, i.e., for every $x \in X$, $\kappa(x)$ is an element of the fiber over x. For example, in a special case of a vector bundle, the zero section is defined when $\kappa(x) = 0$. The value of κ is sought in each fiber of the vector bundle which looks like a horizontal line – cross-section of a fiber bundle. A section of the fiber bundle means to specify the exact fiber state in each fibers of the bundle through local gauge transformation. These local sections can be extended to global sections only if the bundle is trivial. The local sections in non-trivial spaces cannot be globally extended: $LC - GI$.

The fiber bundle structure has also an equivalent description in terms of covering spaces. The covering spaces of X with fiber F are classified by the *action* of fundamental group $\pi(X)$ on F, i.e., $\pi(X) \to Aut(F)$. F consists of a set and $Aut(F)$ is the group of permutations of this set which constitutes E. The fiber bundle has two components: the group of permutations of states of F and its interplay with X. The interplay is the essence of Grothendieck's Galois theory. It gives a way to reconstruct the fundamental group of X as a group of transformations of the universal covering spaces which is same as the automorphisms of the fiber over X.

B Discrete Morse Theory:

Discrete Morse theory (DMT) over a finite simplicial complex $\mathcal{K}$, with $\mathcal{S}_p$ the simplices of dimension p. Discrete Morse function f assigns a single real number to each simplex of $\mathcal{K}$. A function $\mathsf{f}:\mathcal{K}\to\mathbb{R}$ be a discrete Morse function iff, for every d dimensional simplex $\sigma^d\in\mathcal{K}$, the following two conditions hold: $|\{\tau^{(d+1)}>\sigma^{(d)}\mid \mathsf{f}(\tau^{(d+1)})\leq \mathsf{f}(\sigma^{(d)})\}|\leq 1$ and $|\{\tau^{(d-1)}<\sigma^{(d)}\mid \mathsf{f}(\tau^{(d-1)})\geq \mathsf{f}(\sigma^{(d)})\}|\leq 1$. σ is called a critical simplex if such conditions don't hold. The function decreases from 2-simplex to one simplex and vice-versa in reverse direction. Forman's beautiful papers [17] are the standard references on this subject. There is a relation between critical simplices and collapsibility of simplicial complex. If there is no critical simplices in an interval say $[a,b]$ then it is collapsible to a vertex and the space is simply connected. Moreover, the number of critical points of some index are responsible for the topology of the underlying structure. However, one is unable to determine the efficiency of this discrete Morse function with respect to collapsibility i.e; there may be simplices that are critical that could be collapsed while still preserving the homotopy type. A simple example is that of a sphere. So, it is not only critical simplices that are significant in determining the collapsibility of topological space but also the paths between simplices. For example, if there is only one gradient path between two simplices then reversing the direction of the path cancels criticality of both the simplices. The criticality in $\mathcal{K}$ induces non-triviality in paths up to homotopy. Direction (arrows) over simplicial complex can be defined by introducing the concept of combinatorial vector field. It gives direction to each of the simplicial complex. A combinatorial vector field is a map $V:\mathcal{S}\to\mathcal{S}\cup\{0\}$. Given such a map V and $\sigma\in S$ with $V(\sigma)\neq 0$ we draw an arrow on $\mathcal{K}$ whose tail begins at σ and which extends into $V(\sigma)$. It satisfies some properties like if V implies that σ is always face of $V(\sigma)$ then an arrow is possible. If τ is the head of an arrow then it cannot be a tail as well. Moreover every simplex is head and tail of, at most, one arrow. This field classifies each simplex in pairs and there are three disjoint possibilities viz., σ is the head of an arrow ($\sigma\in Image(V)$), σ is the tail of an arrow ($V(\sigma)\neq 0$) and σ is neither head nor tail of any arrow ($V(\sigma)\neq 0$) and $\sigma\notin Image(V)$ (critical simplices).The non-critical simplices occur in pairs $\sigma^{(d)}\subset\tau^{(d+1)}$ where $\mathsf{f}(\sigma^{(d)})\geq\mathsf{f}(\tau^{(d+1)})$. It can be illustrated by drawing an *arrow* from $\sigma^{(d)}$ to $\tau^{(d+1)}$. The points that are neither heads nor tails of any arrow are exactly the critical simplices. A combinatorial vector field V on $\mathcal{K}$ is a collection of pairs $\sigma^{(d)}\subset\tau^{(d+1)}$ of simplices of $\mathcal{K}$ such that each simplex belongs to at most one pair of V. If α is not critical then there is a unique edge $e>\alpha$ with $f(e)\leq f(\alpha)$ i.e., $V(\alpha)=e$ with chosen orientation. If α is critical $V(\alpha)=0$. Given a vector field V on a simplicial complex K, a V-path is a sequence of simplices $\sigma_0^{(d)},\tau_0^{(d+1)},\sigma_1^{(d)},\tau_1^{(d+1)},\sigma_2^{(d)},.......,\tau_r^{(d+1)},\sigma_{r+1}^{(d)}$ such that for each $i=0,....,r$, $\{\sigma,\tau\}\in V$ and $\tau_i>\sigma_{i+1}\neq\sigma_i$. We say such a path is a *non-trivial closed path* if $r\geq 0$ and $\sigma_0=\sigma_{r+1}$. If V is a vector field on a simplicial complex K, V is the gradient vector field of some discrete function on $\mathcal{K}$ if and only if there are no non-trivial closed V-paths i.e., Hasse diagram is acyclic and $\mathcal{K}$ is *collapsible.* If V has (non-trivial) closed paths, then it cannot be the gradient of a function. It corresponds to the transformation of the fundamental group of topological space.

C Stepwise Stratagem

A general stepwise stratagem to apply results of Section 3 involves two broad steps: construction of a discrete topological space and its associated state space based on the given information of possibilistic table of empirical models representing TIM computational framework.

Topological environment The discrete topological space is constructed in the following steps:

1. A simplicial complex $\mathcal{K}$ is constructed from the measurement set of the possibilistic table by considering its power set which is equipped with a discrete topology. $\mathcal{K}$ provides all possible ways in which the basic set-up of the table can be performed.
2. $\mathcal{K}$ constructed from the first step is unable to distinguish between the same contexts because there is a set of same contexts in its combinatorial description. It would be difficult to recognise a specific context among this set which could be turned critical in the next step. To distinguish the contexts we enumerate $\mathcal{K}$ in a way so as to identify its boundary in topological sense.
3. Apply DMT on $\mathcal{K}$. It categorises $\mathcal{K}$ as non-possibilistic, possibilistic and LC-GI corresponding to critical simplices, generic simplices and non-trivial loops respectively. Loops are avoided in DMT but the possibility arises due to equidistant critical simplices.

4. Not all non-possibilistic events can be expressed in the third step because DMT constrains the way different contexts $\mathcal{K}$ are connected. So, iterate the space by constructing simplices until all of the non-possibilistic and LC-GI events are expressed. It is a brute force exhaustive approach.

PTM Computation The TIM computational framework addresses the permutation of the state (fiber) space and consists of the following steps:

1. A regular polyhedra is constructed based on measurement set which represents its different permutations in the state space. Each symmetry of the polyhedra corresponds to a permutation of its measurement set which is represented as its associated polyhedra group.
2. The next step is to find an irreducible representation of polyhedra group which represents its symmetries as invertible matrices forming a group.
3. The valid contexts considered are based on the contexts in the irreducible representation of the group. Any context which belongs to CS of the topological environment is not taken into account because it may not be in the set of irreducible contexts of the fiber space. Both computation and structure constrain each other.
4. Since the fiber space is attached with the base space, the computation (transformation) over the fiber space depends on the type of simplices of its underlying space. Schur lemma for irreducible representation of groups express feasible, infeasible and contextual computation based on $\mathcal{K}$.

D Supplementary Material of Empirical Models

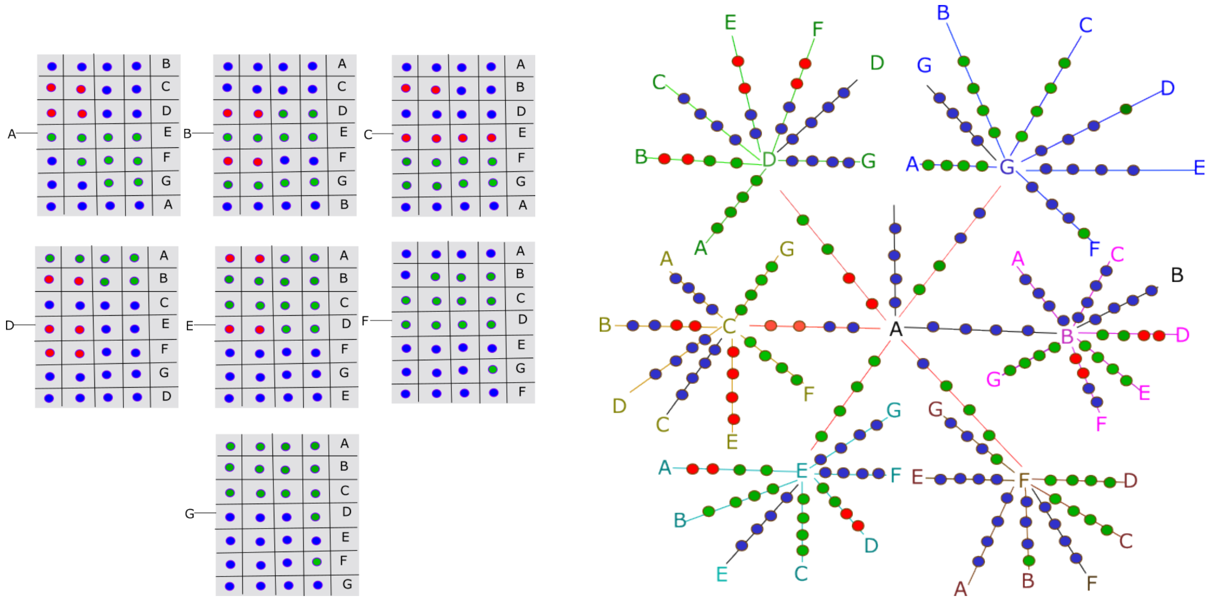

Figure 31: The figure represents all possible permutations of the Kochen-Specker model. There are seven vertices of the pentagonal bipyramid each of which can be permuted with other six vertices and to itself as well as each permutation can have four possibilities in the fiber space represented with four (coloured) circles over each permutation as shown in the left side of the figure. The red and blue coloured circles represent critical simplices and virtual loops which are given from its discrete space of Figure 28 and green coloured circle is the possibilistic/feasible permutation. On the right side of the figure is a clear view of all possibilities which looks like a big flower consisting of seven petals corresponding to permutations of seven vertices. Note that a local section is a transition between any green coloured circles of any petals and the global section is the loop that should start from a green coloured circle of a specific petal and reach back to it. It is clear that there are local sections but there is no family of global sections. Notice, the permutations of vertices to itself (last column of each table) is not green (feasible) which means whichever path one chooses it can never form a loop to the same petal; hence, no possible family of global sections.

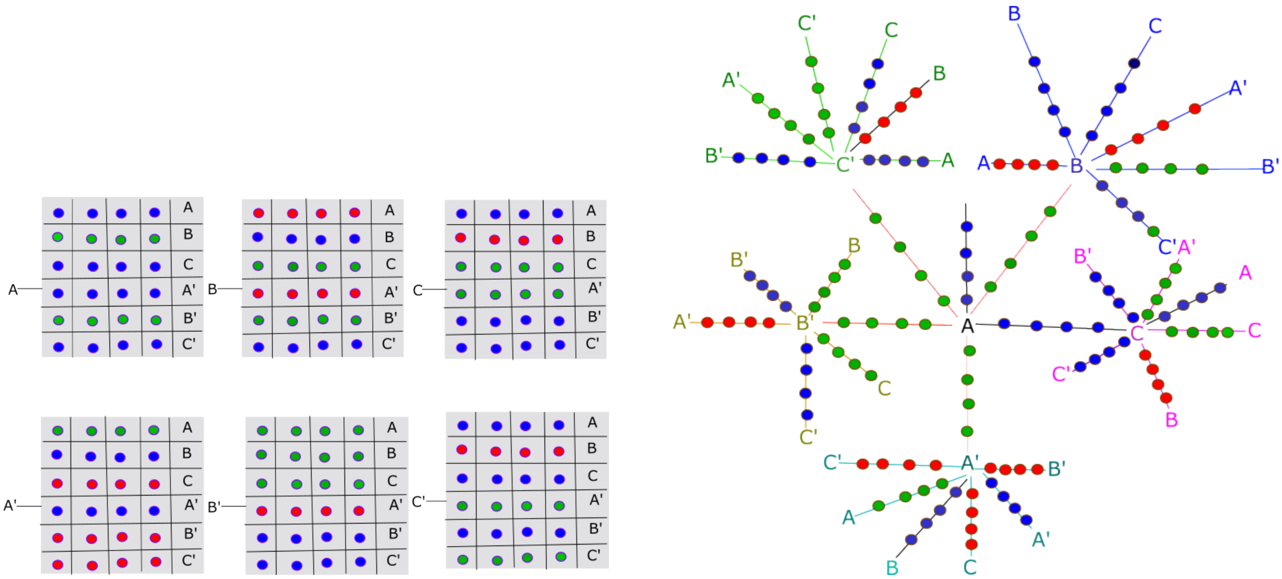

Figure 32: The figure represents all possible permutations of the Greenberger-Horne-Zeilinger Model. There are six vertices of the octahedron each of which can be permuted with other five vertices and to itself as well as each permutation can have four possibilities in the fiber space represented with four (coloured) circles over each permutation as shown in the left side of the figure. The red and blue coloured circles represent critical simplices and virtual loops which are given from its discrete space of Figure 25 and green coloured circle is the possibilistic/feasible permutation. On the right side of the figure is a big flower with six petals. All of the petals do not allow permutations to itself which obstructs the existence of the family global sections except for contexts C and C$^{'}$. In order for these particular contexts to form a global section it should first go through its all local sections and then transit to other petals which is not possible because both contexts have only two feasible fibers (with green coloured circles) and other four are infeasible. Moreover, if we avoid this restriction and allow any possible inter-combinations, these two contexts cannot take into account all the contexts to form a global section because other four petals of flower do not allow any transition; hence no possible and relevant family of global sections.

E Tables of Empirical Models

	(0,0)	(0,1)	(1,0)	(1,1)
(a,b)	1	0	0	1
(a,b')	1	0	0	1
(a',b)	1	0	0	1
(a',b')	0	1	1	0

Contexts	(1,0,0)	(0,1,0)	(1,0,0)
$\{A,B,C\}$	a	b	b
$\{B,D,E\}$	b	b	a
$\{C,D,E\}$	b	b	a
$\{A,D,F\}$	a	b	b
$\{A,E,G\}$	a	a	a

	000	001	010	011	100	101	110	111
ABC	1	0	0	1	0	1	1	0
$AB'C'$	0	1	1	0	1	0	0	1
$A'BC'$	0	1	1	0	1	0	0	1
$A'B'C$	0	1	1	0	1	0	0	1

Figure 33: Upper left is the PR Box table, upper right is Kochen-Specker table where $a = 1$ and $b = 0$ and at the lower center is the Greenberger-Horne-Zeilinger table. The structure of table of Peres-Mermin Square is very differently but we already know the discrete space associated with this model is a torus [23]; as a result, we exempt its construction.

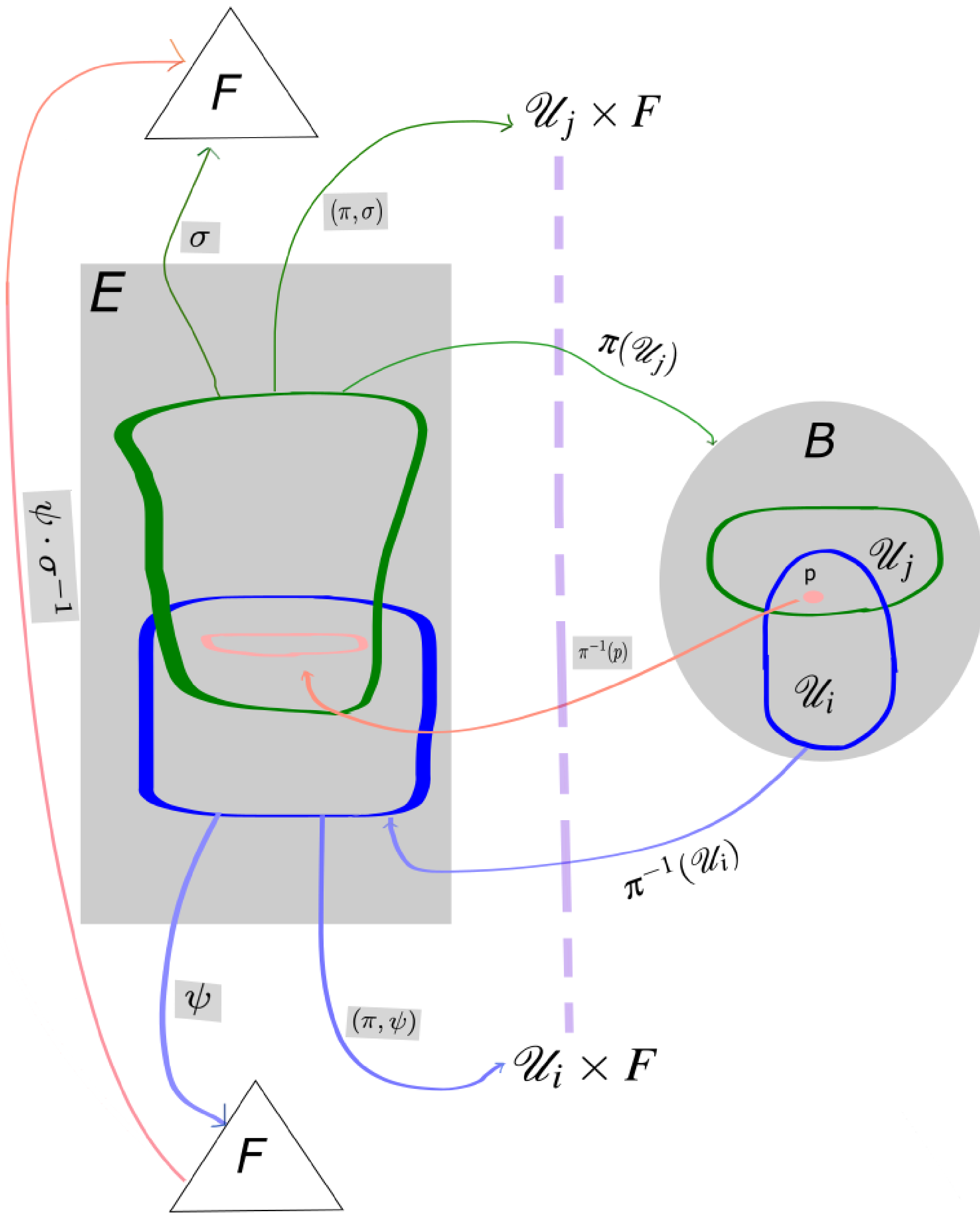

Figure 30: Fiber Bundle

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