Topology of Empirical Models

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Abstract

Abramsky and Branderburger put forth a sheaf based interpretation of non-locality and contextuality as obstructions in inter-knitting local sections together to form a compatible global section. The synergistic interaction of a local function(s) in a global topological space unfolds novel behaviour emerging as a locally consistent but globally inconsistent pattern in data characterising this empirical phenomenon. We explore a general method to associate an explicit approximation of a topological space represented as a simplicial complex to the empirical models, witnessing different strengths of contextuality; alongside their polyhedral description whose symmetries are subject to this structural constraint akin to sheafification based on information of tabular representation of these models. A local consistency corresponds to any possible polyhedral symmetry whereas a global consistency characterises symmetries that take a polyhedron back to itself subject to spatial constraints whose discrete representation quantifies contextuality through strong collapse as (non)existence of critical simplices and virtual loops using discrete Morse theory. The transient virtual loops characterise contextuality as a topological phase transition – a change in homotopy class – that turns relations locally consistent for an observer but globally non-extendable. We apply the framework on several empirical models in the foundation of quantum physics. The framework could provide a practical way to propose new models for witnessing higher dimensional contextuality in guiding physical experiments and linking the phenomenon to the evolution of geometric structures on 3-manifold theory. We provide two new basic models as examples to conceptualise the reverse of our framework of reproducing possibly higher dimensional tables from a given space and associated polyhedron that could lead to observation of new strength of hyper-contextual scenarios. A seven dimensional Mermin-Ardehali-Belinskii-Klyshko model with its graph-based description could be a first step to discover new structures on 3-manifolds for higher dimensional contextuality.

1 Introduction

Quantum contextuality is one of the fundamental non-classical attributes of quantum physics that imparts quantum computing with higher computational power than classical computing. The phenomenon of quantum contextuality is a source of an extraordinary feature of entanglement serving as a theoretical bedrock for the second quantum revolution. The feature is proven real in physical experiments and used as a resource in quantum algorithms to achieve quantum advantage. It advocates the impossibility of context-independent classical descriptions of the prediction of quantum theory.

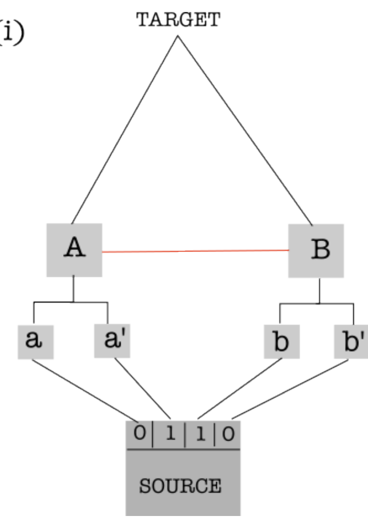
The major role played by contextuality has been the motivation to study more general structures of this feature that are independent of the formalism of quantum mechanics. In 2011, Abramsky and Brandenburg proved that the phenomenon of quantum contextuality classically corresponds exactly to obstructions to the existence of global sections responsible for local consistency and global inconsistency [3]. The new approach is topological that gives passage from local to global and exploits the powerful methods of sheaf theory and cohomology [5]. This emerging semantics has initiated a step towards a general theory of quantum contextuality and a wide range of applicability to the theory of relational database [2], natural language semantics [6], robust constraint satisfaction [4], logic [1] and computation [12].

We explore a symmetry-based framework to quantify non-locality and contextuality in a sheaf-modelled empirical model of quantum physics. It would require an explicit description of a topological space over which an algebraic type resides along with their synergistic interaction akin to sheafication in a symmetry-based description. First, we give a generalisation to associate an explicit approximation of a topological space realised as a simplicial complex to the empirical models quantifying contextuality as (non)existence of critical simplices and transient virtual loops using discrete Morse theory. The transient virtual loops characterise contextuality as a topological phase transition – a change in homotopy class – that turns local relations consistent for an observer but globally non-extendable. These virtual loops are exactly the short loops that arise pervasively in percolation theory of statistical physics, dynamical systems, multilayer network and critical topological phase transition in condensed matter physics. The loops arise in real-time evolving networks responsible for non-linear correlations through its multi-layered architecture that cannot be dealt through a locally tree-like structures [8]. Second, we give a polyhedral description of empirical models whose symmetries are subject to these topological constraints. A local consistency corresponds to any possible polyhedral symmetry whereas global consistency characterises symmetries that take a polyhedron back to itself in the light of associated topological space. We also explore reverse of our framework for two new models, i.e., reproducing tables, possibly of higher dimensions, from a given topological space and its associated polyhedron that expresses new strength of hyper-contextual scenarios, along with guiding physical experiments for quantifying higher dimensional contextuality allowing novel relation to the evolving structures on n-manifold theory. A seven dimensional Mermin-Ardehali-Belinskii-Klyshko model with its graph-based description could be a first step to discover new structures on 3-manifolds for higher dimensional contextuality [13].

2 The Setting

Given two agents A and B, which are space-like separated over a distributed network; both agents have two local bit registers: A with a and $a^{'}$; B with b and $b^{'}$; each of them can store values either 0 or 1. When the registers receive a value (0 or 1) from some (hidden) source in the background, the agents observe and transmit it to some target. For example, A can *choose* its register $a^{'}$ and *observe* 0 and simultaneously B can *choose* b and *observe* 1, which comprises basic event of the system. The frequency of similar events represent the probability of each event as shown in the table on right side along with its basic set-up on left side of Figure 1. Note that each probability p_i is greater than zero.

(i)



(ii)

	00	01	10	11
ab	p_1	p_2	p_3	p_4
ab'	0	p_5	p_6	p_7
a'b	0	p_8	p_9	p_{10}
a'b'	p_{11}	p_{12}	p_{13}	0

Figure 1: (i) is the basic set-up along with its probability distribution table of all events in (ii)

The entry in the table means, if A looks at a and B looks at $b^{'}$, then the probability that A sees 0 and B sees 1 is p_5. $\{a, a^{'}, b, b^{'}\}$ is the measurement set, $\{(a, b)$, $(a, b^{'})$, $(a^{'}, b)$, $(a^{'}, b^{'})\}$ are the measurement contexts, in short contexts, and 0 and 1 are the outputs observed.

Instead of probabilities, one can talk about possibilities of events, i.e. if the table has some probability p_i (greater than zero) between any contexts, then we assign 1 to the corresponding entries of the table, otherwise 0, as shown in table (here, of the Hardy model) on the left of Figure 2. The table has a geometric realisation; with (square) base representing the contexts associated with fibers which represent outcomes; the bundle diagram, as shown in the right side of Figure 2. Its corresponding planar (linearised) diagram is shown in Figure 3 for a better view. Note that in general the square base representing contexts can be realised as a simplicial complex $\mathcal{K}$.

A,B	(00)	(01)	(10)	(11)
(a,b)	1	1	1	1
(a,b')	0	1	1	1
(a',b)	0	1	1	1
(a',b')	1	1	1	0

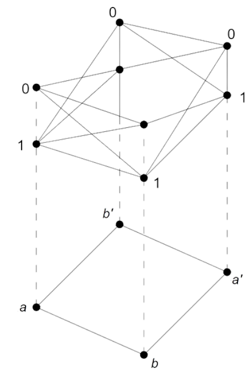


Figure 2: On the left is possibilistic table and on the right is its corresponding bundle diagram: base consisting of contexts and fibers consists of outputs.

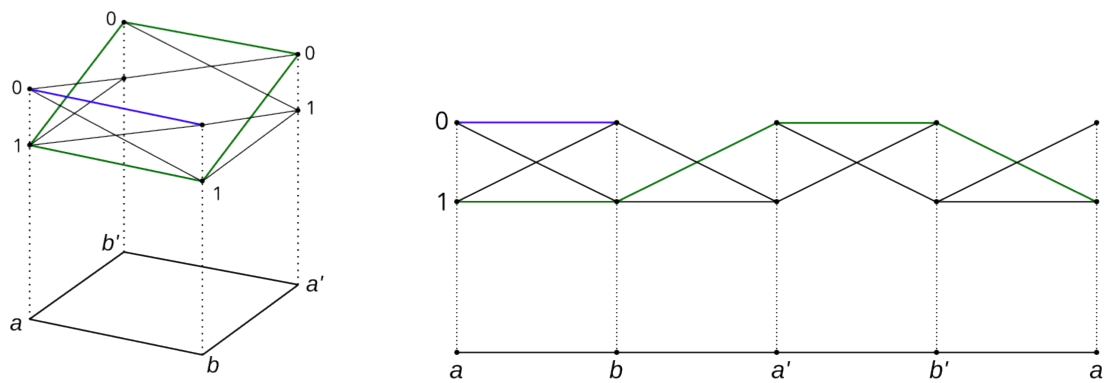


Figure 3: Planar diagram of Hardy Model.

The base of the bundle diagram is a rectangle, a simplicial complex in more general scenarios; consisting of the measurement set $\{a, a^{'}, b, b^{'}\}$ and the fibers consist of possible outputs i.e. 0 and 1. To each context of the measurement set, one assigns its two output values (0 and 1) as a fiber associated with it. Based on any context from the table, there is an edge between their corresponding outputs in the fibers, if and only if, it is a possibilistic event with value 1 in the table. For example, if one chooses context $(a^{'}, b^{'})$ at (11), i.e, output of $a^{'}$ is 1 and that of $b^{'}$ is 1 from table in Figure 2, then there is no edge between their corresponding fibers because the event is a non-possibilistic event as shown at corresponding entry of the table. One closes edges between values in fiber if they can appear together as possible joint outcome over $\mathcal{K}$. For example, the green rectangle in the fiber space of Figure 3 corresponds to four possibilistic contexts; (a, b), (a', b), (a', b') and (a, b') of the table at outputs $(1, 1)$, $(0, 1)$, $(0, 0)$ and $(1, 0)$ respectively. It means, all the contexts are simultaneously satisfiable at these outputs which corresponds to a *closed loop* in the fiber space that correspond to simply-connected (acyclic) underlying simplicial complex. Collapsible property of a space is a necessary and sufficient condition for quantification of contextuality. This property of evaluating consistent families of contexts and their extension stems out from Vorob'ev theorem.

There is one context (a, b) in the table which is locally possible at (00) but globally not possible, i.e. when we extend it through different contexts over the fiber space denoted in blue in Figure 3, it doesn't form a closed loop because the contexts over its base space $\mathcal{K}$ are not simultaneously satisfiable which is expressed as non-collapsibility of $\mathcal{K}$. The table expresses three class of behaviour; possibilistic behaviour, non-possibilistic behaviours and the $LC - GI$ behaviour.

The geometric account in Figure 2 naturally fits the structural description of sheaves. Sheaves comprise of two components that operate in a synergistic manner: a horizontal *topological* component over which its vertical *algebraic-type* resides. A basic example is sheaf *of* vector space *on* abstract simplicial complex. The idea is to move between vectors in each vectors space, which is an individual sheaf or stalk, via linear maps in the vertical component together moving from higher simplices to 0-simplex on its horizontal topological component. The output space of empirical models is of algebraic type represented as a set or generally a ring, and contexts as a topological space in this sheaf structure. We would refer the main paper of the sheaf modelling of empirical models of Abramsky [3] and its cohomological treatment [5] for detailed account of this algebraic machinery.

3 Methods

3.1 Strong Collapse and Contextuality

Collapsibility of a simplicial complex quantifies the concept of contextuality. The property implies that the values, algebraic-type such as set, assigned to variables, structured as a simplicial complex, are simultaneous-measurable, characterising essence of quantum contextuality. We provide a general method to associate an explicit approximation of a topological space to empirical models of quantum physics based on information provided in their tables using discrete Morse theory [11]. A very short account on DMT is also provided in Appendix B. The collapsibility measure in this theory is characterised in terms of critical simplices (CS), i.e., non-existence of CS of a simplicial complex imply collapsibility. A key observation in empirical models is that the experimenter does not have a priori final structural description of the space but rather collects data at each round of the experiment to construct the space as an evolving structure analogous to topological data analysis. There is an emergence of short loops, here to be called as virtual loops (VL), in evolving real world networks that are pervasive in percolation theory of statistical physics, dynamical systems, multilayer networks and critical phenomenon in condensed matter physics. These transient virtual loops are responsible for non-linear correlation in a multi-layered architecture, like a simplicial complex, characterising a topological phase transitions and fail to be dealt through a locally tree-like approximations [8]. As a result, we have another condition for collapsibility which is non-existence of virtual loops along with already stated empty CS-set implying contextuality. The non-existence of CS and VL in a simplicial complex is referred as a *strong collapse*.

The virtual loops do not violate causality and the definition of discrete Morse function $\mathfrak{f}$ because these transient loops vanishes when one reiterates $\mathfrak{f}$ over the final $\mathcal{K}$ that evolved after going through several intermediate $\mathcal{K}$'s. For instance in the example of the Liar cycle model as discussed in Subsection 3.2, the final $\mathcal{K}_7$ is reproduced after going through intermediate $\mathcal{K}_3$, $\mathcal{K}_4$, $\mathcal{K}_5$, $\mathcal{K}_6$, based on the information of its associated table. One can reiterate $\mathfrak{f}$ on the structure at each level of iteration to yield a virtual-loop free discrete manifold after the structure has gone through a topological phase transition. In our case, this phenomenon of topological transition pervasive in critical phenomenon of condensed matter physics due to these short loops is characterised as a change in homotopy class of the space. $\mathcal{K}_4$ is homeomorphic to a 2-torus that transforms into $\mathcal{K}_5$ homeomorphic to a 3-torus. The notation n before torus is a 2-dimensional hollow torus. This loop is a boundary dynamics in manifold theory and emerge due to equidistant critical simplices transforming a 2-torus into a 3-torus. The evolution of the space that will be associated to empirical model is discussed in subsection 3.2 and applied to several models in Section C. The phenomenon this boundary dynamic and union of surfaces is notably conceivable in 3-manifold theory, a simple example could be toral decomposition (JSJ decomposition) of a 3-manifold.

Definition 1. *Let* $\mathbf{e}$ *be an empirical model and* $\mathcal{K}$ *its associated simplicial complex. The non-possibilistic events of* $\mathbf{e}$ *are represented by critical simplices and locally consistent-globally inconsistent incompatible family of sections by class of transient virtual loops.*

Definition 2. *A strong collapse of a simplicial complex is the non-existence of critical simplices and virtual loops.*

Theorem 1. *If a simplicial complex* $\mathcal{K}$ *is not strong collapsible then* $\mathcal{K}$ *is contextual in nature.*

Proof. VL cannot be resolved using tree-type approximation and introduce cycles in $\mathcal{K}$ and CS is a criteria of non-collapsibility in discrete Morse theory. The existence of (CS, VL) in $\mathcal{K}$ is non-collapsibility to $\{\emptyset\}$ which implies contextuality.

□

Definition 3. *A polyhedral-description of an empirical model is the polyhedron that could be associated with the model whose number of vertices are equal to cardinality of the measurement set.*

The polyhedral-description induces symmetry transformation between different contexts of the model, ie., provides a way to permute any contexts. The symmetries are subjected to the structural constraint from associated simplicial complex of the model.

Definition 4. *A local consistency corresponds to any possible polyhedral symmetry whereas global consistency characterises symmetries that take a polyhedron back to itself in the light of associated topological space.*

Theorem 2. *Any empirical model representable by the Liar cycle-type scenario has a topological representation of a surface of genus* 5 *that constrains symmetries of its associated square bipyramid – a polyhedral-description – that turns out to be exhibit strongly contextual behaviour.*

Proof. Proof by construction:topological representation 3.2 and polyhedral-description 10 11 □

Theorem 3. *Any empirical model representable by the trivial-type scenario has a topological representation of a surface of genus* 0 *that constrains symmetries of its associated tetrahedron– a polyhedral-description – that turns out to be exhibit no contextual behaviour.*

Proof. Proof by construction in Subsection 21. □

Theorem 4. *Any empirical model representable by the Bell-type scenario has a topological representation of a surface of genus* 1 *that constrains symmetries of its associated tetrahedron– a polyhedral-description – that turns out to be exhibit logically contextual behaviour.*

Proof. Proof by construction in Subsection C.2. □

Theorem 5. *Any empirical model representable by the Hardy-type scenario has a topological representation of a surface of genus* 4 *that constrains symmetries of its associated square bipyramid– a polyhedral-description – that turns out to be exhibit logically contextual behaviour.*

Proof. Proof by construction in Subsection C.3. □

Theorem 6. *Any empirical model representable by the tetrahedron-type scenario has a topological representation of a surface of genus* 8 *that constrains symmetries of its associated tetrahedron– a polyhedral-description – that turns out to be exhibit strongly contextual behaviour.*

Proof. Proof by construction in Subsection C.4. □

Theorem 7. *Any empirical model representable by the KCBS-type scenario has a topological representation of a surface of genus* 5 *that constrains symmetries of its associated square bipyramid– a polyhedral-description – that turns out to be exhibit strongly contextual behaviour.*

Proof. Proof by construction in Subsection C.5. □

Theorem 8. *Any empirical model representable by the Svetlichny's box-type scenario has a topological representation of a surface of genus* 8 *that constrains symmetries of its associated tetrahedron– a polyhedral-description – that turns out to be exhibit strongly contextual behaviour.*

Proof. Proof by construction in Subsection C.6. □

We are given the possibilistic tables of the empirical models containing information about the (non)possibility of events corresponding to measurement contexts and their possible outputs. The entry of the table is 1 for possibilistic events and 0 for non-possibilistic events. Our goal is to understand the phenomenon of contextuality in terms of symmetry. The synthesis of our approach is quantification of symmetry transformations in a structural constraint. A discrete topological space is constructed iteratively from the given information of the possibilistic tables of the empirical models. This space constrains the symmetry transformations of a polyhedron whose vertices correspond to possible contexts of the model, i.e., the cardinality of the measurement set is the vertex set of the polyhedron. For instance, tetrahedron is the polyhedral description of the Bell model because cardinality of measurement sent is 4. The description allows permutation between different contexts to quantify different class of symmetries. The symmetry transformations of the polyhedron representing possible contexts is subjected to a structural constraint of the discrete space. The local consistency corresponds to any local possible permutations of the polyhedron and the global consistency means transformations that takes polyhedron through a series of transformations back to itself. There are strongly contextual models that have no family of such global transformations. The framework is general for any empirical models and we provide examples of Liar cycle model, Klyachko-Can-Binicioglu-Shumovsky model (KCBS), Hardy model, Bell model, tetrahedron model and Svetlichny's box model.

3.2 Topology of Empirical Models

We construct a discrete topological space from the information provided in tables of the empirical models as shown in Figure 4. Let us take a simple example of a Liar cycle to clearly explain the construction. Other models follow same method.

	00	01	10	11
ab	1	0	0	1
bc	1	0	0	1
cd	1	0	0	1
de	1	0	0	1
ea	0	1	1	0

Figure 4: Possibilistic table of Liar cycle model.

Our goal is to express the class of behaviour of the tables of the empirical models. It includes possibilistic events, non-possibilistic events and locally consistent and globally inconsistent events (LI-GI) which would correspond to generic simplices, critical simplices and virtual loops respectively in the constructed topological space.

A conceivable first step is to provide a combinatorial description of this model by considering power set of the measurement set $\{a, b, c, d, e\}$ which is equipped with a discrete topology $\mathcal{K}_1$ as shown in (i) of Figure 5. $\mathcal{K}_1$ is a combinatorial description of all possible ways in which the physical experiment can be performed. The set $\{a, b, c, d, e\}$ can be thought of variables that store values either 0 or 1 measured by two agents A and B to observe the corresponding values. A possible set-up would be A chooses a and B chooses b, afterwards A chooses c and B chooses d, followed by A choosing e to observe values either 0 or 1. A possible permutations of such a set-up is also shown in (i) of Figure 5. Contextuality is the impossibility to assign value (0 or 1) to the result of the measurement of a physical property (variables) because it depends on the measurement context, i.e., what other properties are simultaneously measured with it. This description is naturally more informative than tables, i.e, it allows different context to be measured simultaneously like (abb) and similar other diagonal contexts, where A chooses a and B chooses b and afterwards A chooses b in different set-up. These diagonal contexts do not add any significant topological context, i.e, using cell complexes instead of simplicial structure $\mathcal{K}_1$ would avoids these contexts realising same topology and have same Euler characteristics.

It is required to express non-possibilistic events of Liar cycle model along with LC-GI events as a result of these obstructions, but all simplices of $\mathcal{K}_1$ are generic. A random selection of any context(s) to critical in $\mathcal{K}_1$ based on non-possibilistic events in its table would lead to context indistinguishability issue, i.e., there are several same contexts over $\mathcal{K}_1$, for instance, for a non-possibilistic event ea there are almost eight ea$'s$ in $\mathcal{K}_1$. It turns out that selecting any context randomly would also force the agents to select that particular output at that context which violates the axiom of choice in empirical models. For instance, ea at (00) is non-possibilistic event and turning a particular ea critical in $\mathcal{K}_1$ would force agents to observe this particular output at that particular context. It constrains the element of choice of agents and structure $\mathcal{K}_1$ becomes self-fulfilling prophecy to trivially reproduce any behaviour.

It is improper to randomly turn any simplex of $\mathcal{K}_1$ critical based on non-possibilistic events in the table. A background condition is required to systematically turn a set of particular simplices critical. This results in constraint on the maximum number of critical simplices to be included over the structure $\mathcal{K}_1$. It means one cannot include all non-possibilistic events at once over $\mathcal{K}_1$ but rather iteratively add additional discrete spaces until it expresses class of behaviour of the empirical models. We use discrete Morse theory as a background condition to turn a set of simplices critical iteratively for non-possibilistic events along with emerging virtual loops due to these obstructions for LC-GI events. Additionally, the axiom of choice is respected by assigning all possible outputs, both 0 and 1, to each context so that agents have an element of choice to measure any variables and observe any output independently.

$\mathcal{K}_1$ is homeomorphic to a real line or disc topology because the Euler characteristics χ turns out to be 1, vertices (v) = 25, edges (e) = 56 and faces (f) = 32; $\chi = v - e + f = 1$. The cohomology of a disc is trivial and contractible to a point which can only express possibilistic events as in the case of trivial models with no non-possibilistic events requiring only generic simplices offering no constraint on its tetrahedral symmetry transformations i.e., all global sections exist as shown in Section 21. This linear topology recalls local hidden variables that cannot reproduce contextuality due to Bell-Kochen-Specker theorem.

The context indistinguishability issue is also resolved by designating each possible combinatorial set up as levels and then considering all set of possible loops, paths starting and ending at the same vertex, and evaluate the particular context at the particular set-up not contained in the loop-set would correspond to the critical simplex as discussed in Appendix A. The reason to evaluate loops is because semantic of the structure to express contextuality is described by the fundamental group which considers loops in $\mathcal{K}_1$. On contrary discrete Morse theory do not allow loops and one of the natural ways is the identification of the boundary topologising $\mathcal{K}_1$ from a trivial topology to a torus surface, turning $\mathcal{K}_1$ to $\mathcal{K}_2$ as shown in (ii) of Figure 5. The canonical commutation relation between two observables has a algebraic torus representation and reasonable minimal surface to express contextuality. Torus encodes naturally the obstruction led sheafication of contextuality unlike positively curved spaces like sphere where we would be required to understand relation between curvature and contextuality which are not related in sheaf based interpretation. $\mathcal{K}_2$ has $\chi = 0$ homeomorphic to a torus. We apply discrete Morse function $\mathfrak{f}$ over $\mathcal{K}_2$ represented as $\mathcal{K}_3$. $\mathfrak{f}$ assigns to each simplices a natural number $\mathbb{N}$ under the condition that as one moves from lower to higher simplices the number should increase. The identified simplices are assigned same values which constrains the value assignment to a set of simplices where number decreases contrary to $\mathfrak{f}$ are turned critical. It follows that four 0-simplices, two 1-simplices and one 2-simplex of $\mathcal{K}_3$ are turned critical which correspond to four a, contexts cd, de and context bcc respectively. The assignment of $\mathbb{N}$ to each simplex is not shown for brevity. Additionally, a well defined CW attachment map between these critical simplices gives a torus. Notice, there are only two valid contexts cd and de in $\mathcal{K}_3$ that corresponds to two non-possibilistic events of the table. The four single contexts a is not meaningful because contextuality is a phenomenon between at least two observables. The critical contexts in the structure would be validated or invalidated based on the contexts that will be considered in the polyhedral description of the model, i.e., context bcc would be considered only when it will be part of the symmetry transformations represented as square bipyramid as in Figure 11.

The structure $\mathcal{K}_3$ cannot include all non-possibilistic events, for instance, $\mathcal{K}_3$ includes only two relevant non-possibilistic contexts cd and de but there are more other contexts to be included. Moreover, the structure constrains the maximum number of critical simplices included, hence we require to iterate or attach another torus surface to $\mathcal{K}_3$ turning it to $\mathcal{K}_4$ as shown in Figure 6. $\mathcal{K}_4$ includes two more non-possibilistic event de and ea. Upon calculating the χ of $\mathcal{K}_4$ turns to be -2 which means it is homeomorphic to 2-torus. We would require to iterate $\mathcal{K}_4$ to another space until it includes all non-possibilistic events of the table of the model. It passes through many $\mathcal{K}'s$ and eventually adding three more discrete spaces to $\mathcal{K}_4$ expresses iteratively all non-possibilistic events in the table of this model which corresponds to 5-torus space as shown comprehensively in Figure 9. The two blue lines in the figure represent end of one torus and beginning of other torus being attached at the boundary. This blue regime has two equidistant critical 0-simplices, for instance, four red coloured circles named a and b in Figure 6. There is a possibility of transient virtual loops that could form due to equidistant critical points in the discrete space at the boundary of two attached tori represent in blue lines. The area between these blue lines have a choice to go in either directions which facilitate emergence of these virtual loops. These non-trivial loops do not violate the description of $\mathfrak{f}$ as it is definition presumes loop free space as well as do not violate causality because these loops are transient and a possible invariant at the observational level. One can re-iterate $\mathfrak{f}$ over the final space, here 5-torus, and these loops vanish. The virtual loop is a quantification of a topological transition from one homotopy class to another. Conceptually, an experimenter does not have a priori access to the final discrete space but it collects data at each round of the experiment to construct the intermediate spaces similar to topological data analysis. These virtual loops express class of LC-GI events wherein locally the simplices are not critical (infeasible) but globally transient into new topological space turning it globally inconsistent. The weaving of contexts of the topological space associated with the models is iterated in a peculiar fashion. Notice, 1-torus starts with context a and the space is woven by naturally going through the natural order of contexts until it reaches last iteration which starts with e and also other intermediate spaces passing through natural order of permutations. At times we force the ordering of contexts so that every iteration contains a non-possibilistic event and it does not affect the results of our approach, also it is computationally hard task. The constructed topological space can independently express contextuality as non-trivial fundamental group of the space, i.e., non-existence of critical simplices and non-trivial virtual loops imply non-contextuality.

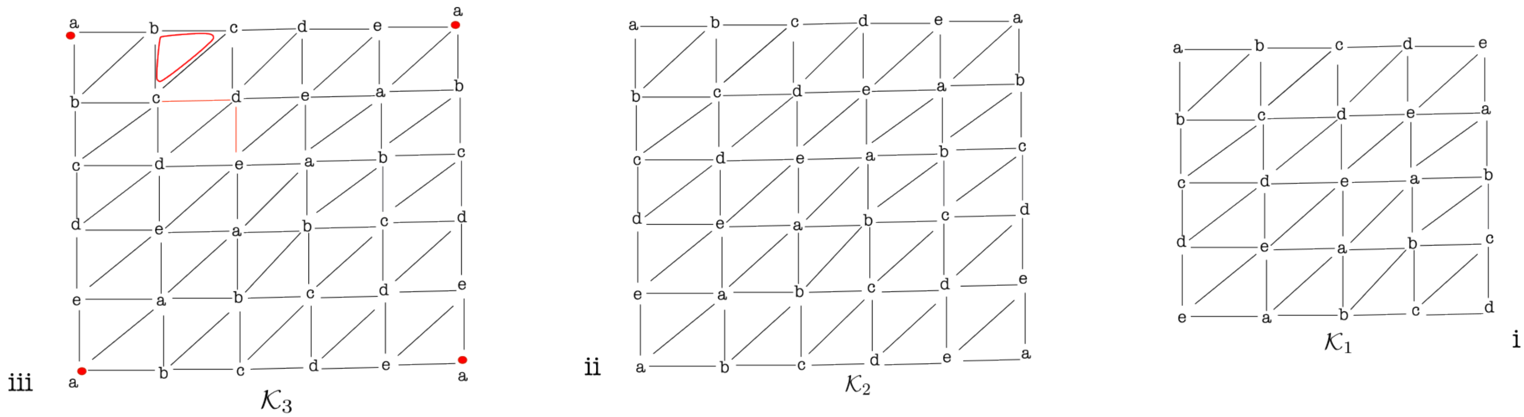


Figure 5: The different steps of iterative construction of discrete space associated with Liar cycle model based on information of its table along with Figure 6, 7, 8, 9 is shown. (i) shows a combinatorial description of the measurement set $\{a, b, c, d, e\}$ which gives different permutations in which an experiment can be performed. (ii) topologises $\mathcal{K}_1$ of (i) by identifying its boundary turning a disc-like trivial topology into a 1-torus $\mathcal{K}_2$ due to context indistinguishability issue and associating element of choice to empirical models. (iii) applies discrete Morse theory on $\mathcal{K}_2$ to yeild $\mathcal{K}_3$, providing a systematic way to characterise non-possibilistic events and LC-GI events. As a result, (iii) contains two critical simplices cd and de based on non-possibilistic events of its table 4. The consideration of other additional simplex bcc will depend on the polyhedral description of the model, i.e., if bcc is a valid permutation of polyhedron, otherwise invalidated.

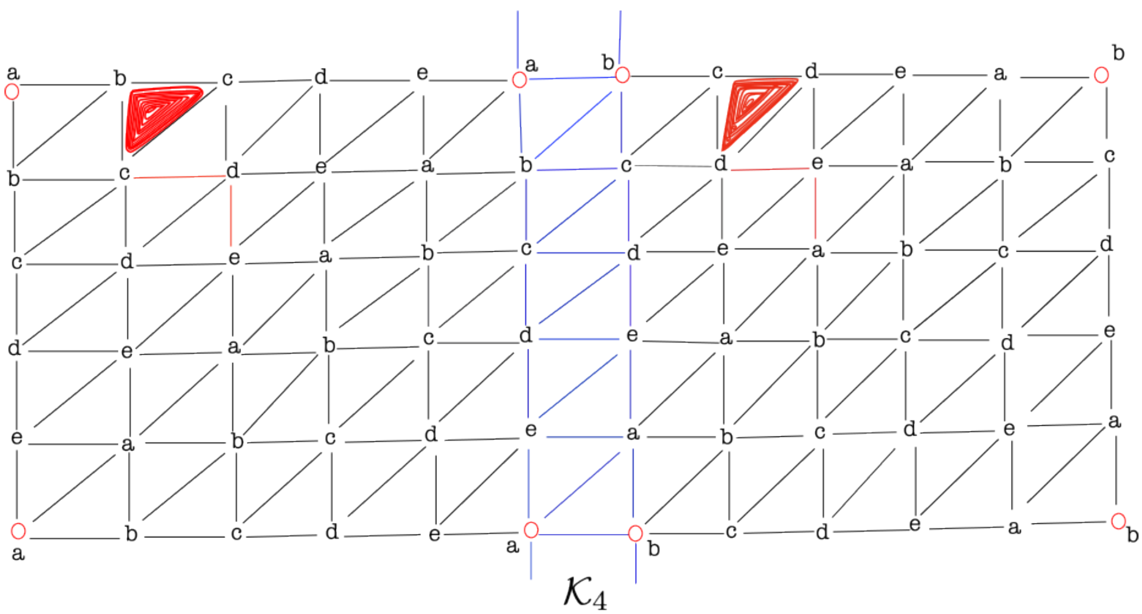


Figure 6: $\mathcal{K}_3$ does not take into account other non-possibilistic events due to constrains from on $\mathfrak{f}$. $\mathcal{K}_3$ is added with another surface changing it to $\mathcal{K}_4$ homeomorphic to 2-torus containing another set of critical simplices de and ea, as well as possibility of emergence of non-trivial virtual loops at the attaching boundary shown in blue colour due to equidistant critical simplices. The colour red represent critical simplices and colour blue represent virtual loops.

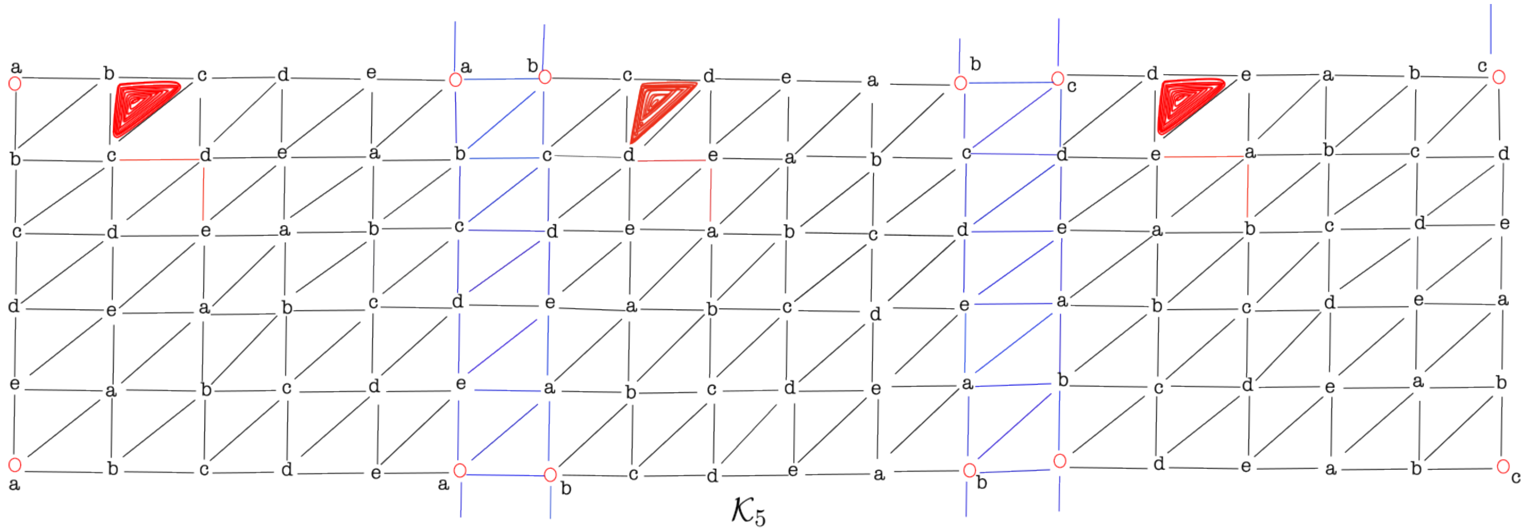


Figure 7: The figure along with Figure 8 are intermediate iterations of final surface $\mathcal{K}_7$. $\mathcal{K}_4$ turns to $\mathcal{K}_5$ homeomorphic to 3-torus containing another set of critical simplices ea and ab and any possible combination of virtual loops represented in blue colour.

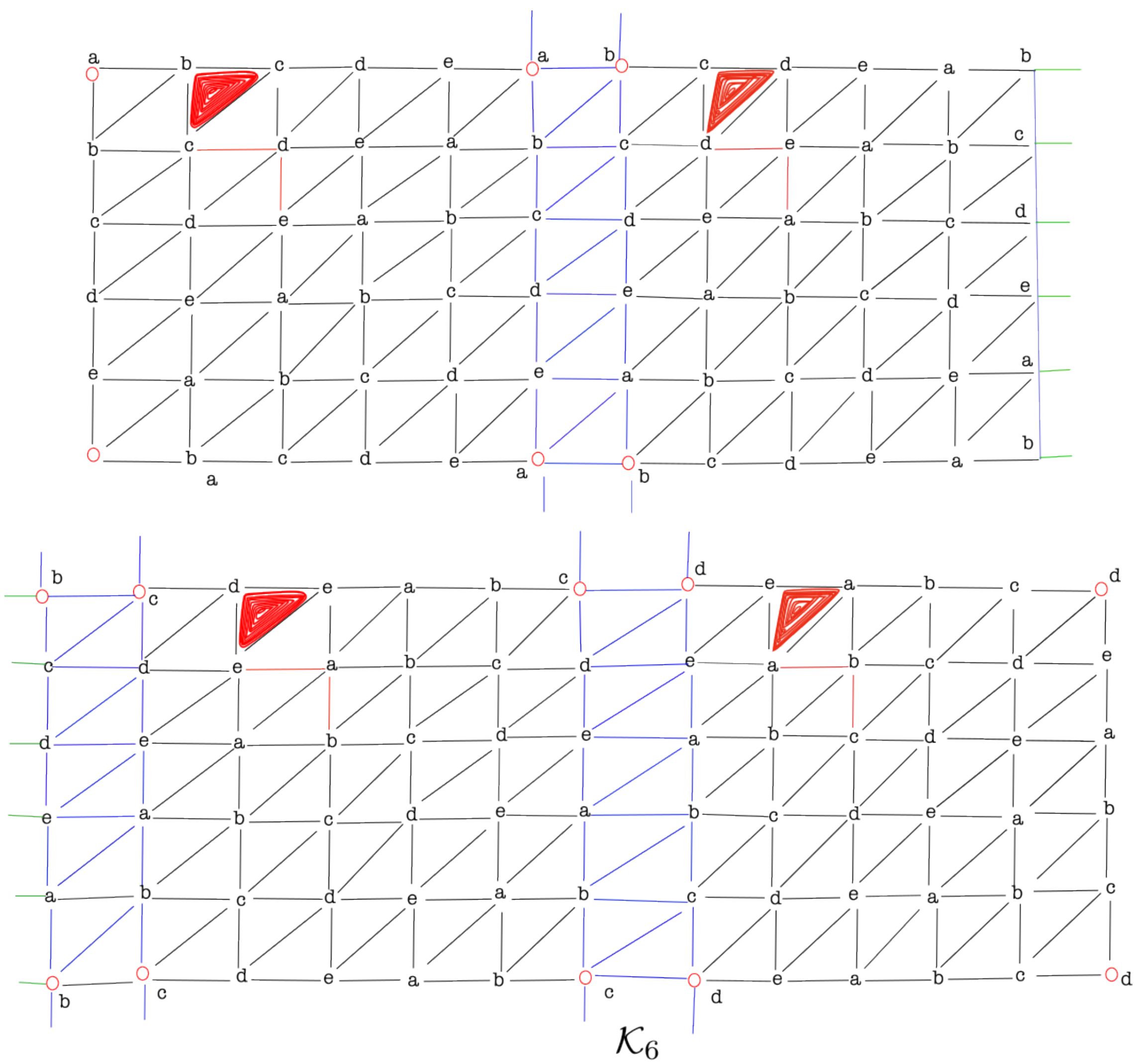
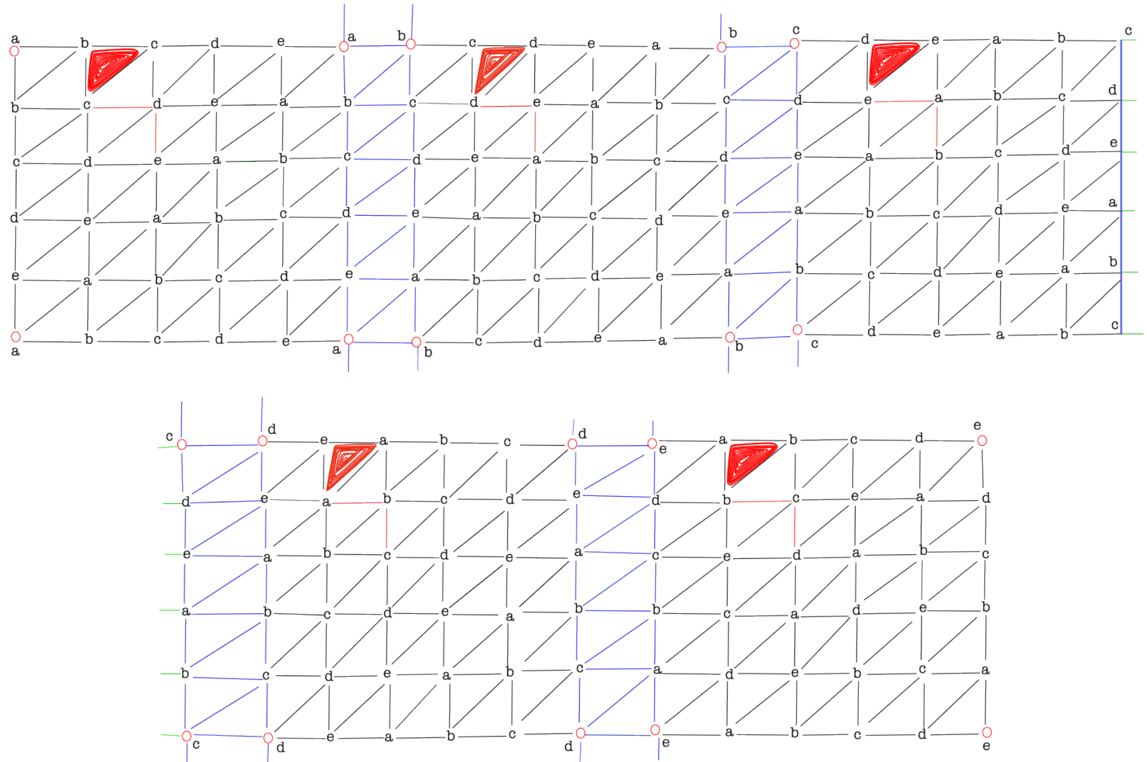


Figure 8: $\mathcal{K}_5$ turns into $\mathcal{K}_6$ to contain another set of critical simplices ab and bc, along with blue coloured possible combination of virtual loops.



$\mathcal{K}_7$

Figure 9: The final iteration turning $\mathcal{K}_6$ to $\mathcal{K}_7$ to contain final set of non-possibilistic events as critical simplices bc and cd along with possibility of virtual loops in blue. The space $\mathcal{K}_7$ expresses all class of possible behaviour of Liar cycle model. This space constrains the symmetry transformations of the contexts via set of critical simplices and set of possible loops to the polyhedral representation of the model as shown in Figure 11. Notice, many critical simplices that are not part of the table 4 are added due to condition of $\mathfrak{f}$ on $\mathcal{K}_7$ in order to allow a systematic assignment of events from its table; along with set of virtual loops (and their different possible linear combinations) in blue are only validated if they are part of a reasonable symmetry operation of its polyhedral description, otherwise ineffective.

3.3 Symmetry and Contextuality

A simplicial complex is a general representation of contexts whose vertices are associated with additional algebraic data that is described through a set (a field or a ring) representing outputs in a sheaf-based modelling of empirical models. A simple geometric description is the bundle diagram as shown in Figure 2. As a result, our method addresses topological space and its associated data along with their synergistic interaction.

The first part of our method above constructed a simplicial complex that approximates a maximum surface expressing class of behaviour of the Liar cycle model as in Figure 9. Now, the next part is the quantification of all possible permutations of paths over fiber space which is expressed through a polyhedral-description of contexts and their associated fibers as shown in Figure 11.

The measurement set can be described as a polyhedron whose vertices equals to cardinality of the set itself and represent all possible contexts of the model. The description allows permutations between contexts which is constrained by the above constructed discrete space via set of critical simplices and set of possible virtual loops. Each vertex of the polyhedron is associated with all possible outputs represented as a fiber over it. This representation is similar to bundle diagram of sheaf modelling as shown in Figure 2. The square base representing contexts is replaced by a polyhedron to be able to quantify symmetry transformations among contexts. A local consistency corresponds to any possible transformation of the polyhedron and the global consistency corresponds to transformations of the polyhedron that takes it possibly through several transformations and brings it back to its initial context (vertex). The inability to form this loop transformation from a particular context back to itself quantifies contextuality. The description is closely related to sheaf based modelling of empirical models where a loop in the fiber space containing output means their corresponding contexts are simultaneously satisfiable, hence non-contextual.

Measurement set $\mathcal{M} = \{a, b, c, d, e\}$ of Liar cycle model has cardinality ($|\mathcal{M}| = 5$) which corresponds to total vertices of the square bipyramid – a polyhedral description of the model as shown in Figure 11. Each symmetry of a square pyramid corresponds to the permutation of its five vertices representing contexts. Each vertex of this square pyramid is attached with both possible outputs 0 and 1 represented in purple coloured lines of Figure 11, providing each contexts with element of choice. *ab bc cd de ea* are relevant contexts of the model as considered in its table description, but the polyhedral description of model allows most of the possible combinations of the contexts which maximally would be power set $\mathcal{P}$ of $\mathcal{M}$ as:

$\mathcal{P}(\mathcal{M}) = \{\{\emptyset\}, \{a,b,c,d,e\}, \{a\}, \{b\}, \{c\}, \{d\}, \{e\}, \{a,b\}, \{a,c\}, \{a,d\}, \{a,e\}, \{b,c\}, \{b,d\}, \{b,e\}, \{c,d\}, \{c,e\}, \{d,e\}, \{a,b,c\}, \{a,b,d\}, \{a,b,e\}, \{a,c,d\}, \{a,c,e\}, \{a,d,e\}, \{b,c,d\}, \{b,c,e\}, \{b,d,e\}, \{c,d,e\}, \{a,b,c,d\}, \{a,b,c,e\}, \{a,c,d,e\}, \{b,c,d,e\}, \{a,b,d,e\}\}$

The symmetry transformations of the square bipyramid is constrained by the set of critical simplices and possible virtual loops of the discrete space as shown in Figure 9. The set of critical simplices CS are as: $CS =$ {bcc, cd, de, cdd, de, ea, dee, ea, ab, eaa, ab, bc, abb, bc, cd} and set of possible virtual loops VL are as: $VL =$ {abb, bbc, bcc, ccd, ddc, dde, eed, eea, aea, aab, cdd, dde, eed, eea, aea, aab, abb, bbc, bcc, ccd, bcc, ccd, cdd, dde, eed, eea, aae, aab, bba, bbc, dee, eed, eda, adc, abc, bbc, bbc, bca, acd, ade }. VL-set can also contain different combinations like that of four context-loops such as: { abcb, abcd, abde, abea, abab, bccd, bcde, bcea, bcab, cdde, cdea, cdab, deea, deab, eaab, bbcc, bbdd, bbee, bbaa, ccdd, ccee, ccaa, ddee, ddaa, eeaa} as formed as a boundary phenomenon of two tori, for instance as in Figure 6 depicting possible virtual loops at boundary represented in blue. There could be other different 4-context possibilities between other boundaries but we neglect here for brevity.

The CS-set and VL-set puts constraint on the possible symmetries of its polyhedral description. A linearised planar diagram of the square pyramid that considers all of its edges representing all possible contexts of the model is shown on right side of Figure 11. We also show only relevant contexts of the model as depicted from its table in Figure 10. A permutation of any contexts is possible if its underlying simplices of background simplicial complex as constructed in Figure 9 allows it, i..e., the simplices corresponding to the contexts is generic represented as black colour in Figure 9. Otherwise, a permutation is not possible if their corresponding simplices are elements of critical simplices set and virtual loop set represented in red and blue respectively. The possible family of global sections are as: {abcdea, bcdeaceb, cdeac, deacebd, eace, acebda}, each starting and ending with same vertex. Allowing all possible permutations to quantify wide range of symmetries of the model unlike only relevant contexts allows local sections in Liar model but there is no family of global section possible as shown in Figure 11.

Remark. *A local section corresponds to any possible permutation of the polyhedron, here square bipyramid represented in its corresponding green colour in linearised planar diagram. A global section corresponds to transformations that permute the context, possibly through several intermediate permutations, back to itself. The description quantifies essential concept behind contextuality, i.e., impossibility of simultaneous value assignment to variables of a quantum system.*

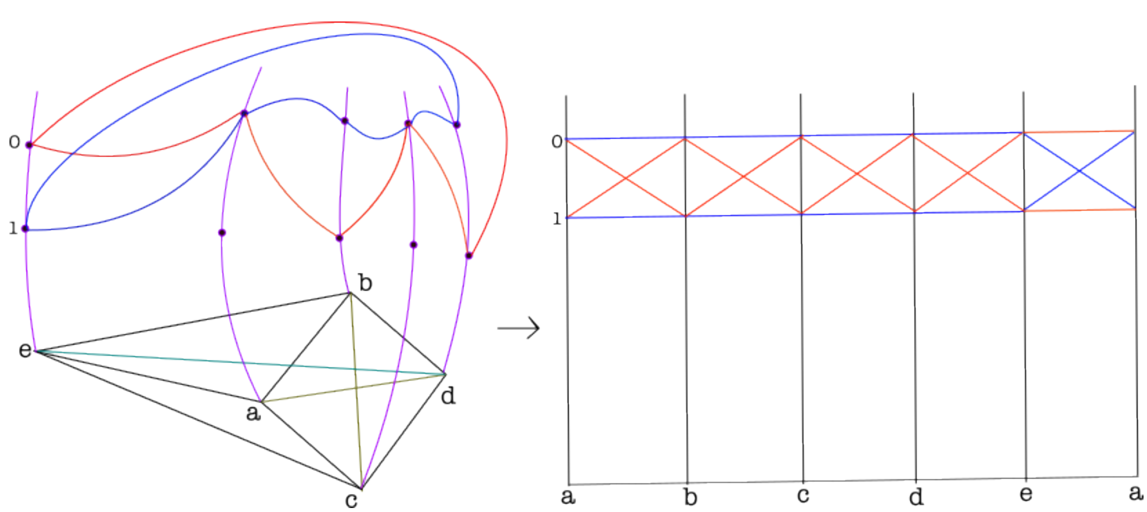


Figure 10: The polyhedral description as square bipyramid of Liar cycle model based on cardinality of its measurement set is shown on left hand side of the figure. On the right hand side is the linearised planar view considering edges of the square pyramid that are relevant contexts in its table, i.e., *ab bc*, *cd*, *de*, *ea*. There is no family of global sections possible due to constraint on the symmetries of square pyramid, and on its represented contexts, from its corresponding discrete space of Figure 9 via CS-set and VL-set denoted in red colour and blue colour respectively, hence model is strongly contextual. An example of infeasible path and LC-GI path represented in red and blue corresponding to non-possibilistic and LC-GI events are shown on left hand side of the figure.

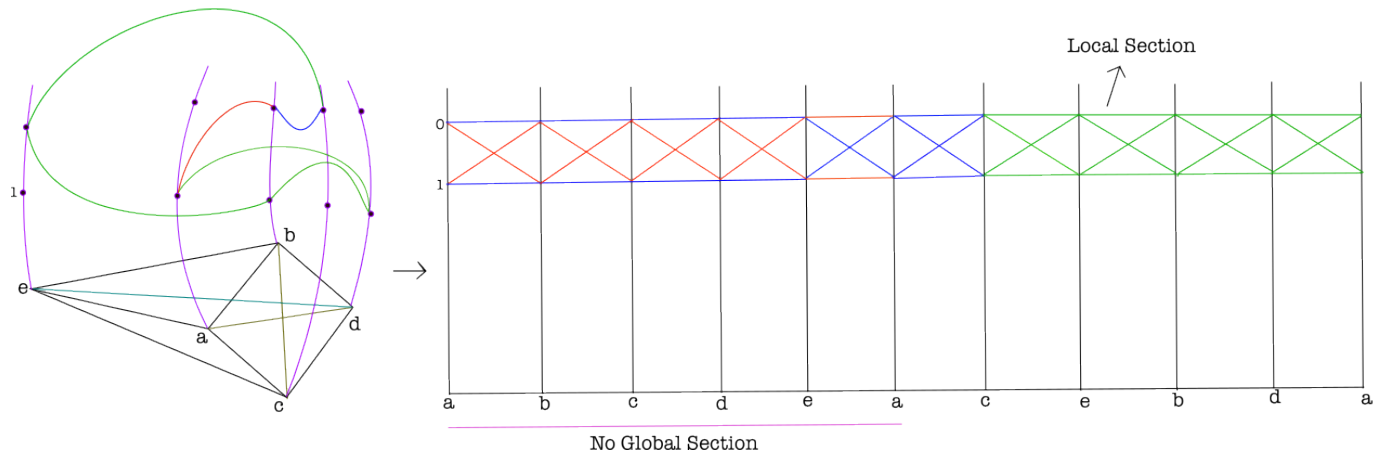


Figure 11: This is also a polyhedral description as square bipyramid of Liar cycle model but represents almost all possible contexts in its linearised planar diagram on the right hand side of the figure unlike only relevant contexts of the model. It allows a wide range of possible permutations between contexts in the presence of the structural constraint of Figure 9. Notice, there are possible local sections denoted in green colour but globally there is no family of sections possible, i.e., there is no possible (global) section that starts with a vertex and ends back to the same vertex as shown by pink line below context starting from a to itself. An example of section (path) that starts from c at 0 goes to e at 0, after goes to b at 1 followed by d and a at 1 and 1 respectively is shown as section in green colour of left hand side of figure. Afterwards in order to be part of compatible family of section, it should come back to c through b, but it does not come back because the path from a at 1 to b at 0 is critical (red) and the path from b at 0 back to c at 0 corresponds to LC-GI events expressed as part of VL-set in blue, hence strongly contextual.

4 Reversing the Approach

The framework in Section 3 associates an approximation of a topological space and its polyhedral description along with their synergistic interaction to express behaviour of empirical models from the information provided in their tables. What about reverse, i.e., given a relevant topological space and polyhedral description, could we construct possibly higher dimensional tables that could be used to construct new empirical models. The synthesis of novel tables could be useful in guiding experiments witnessing higher dimensional contextuality [13] and observation of new scenarios of hyper-contextuality. We provide two examples that constructs table that could possibly correspond to a new empirical model and express hyper-contextuality, i.e., models that have neither any possible local section nor any possible global section unlike the models discussed in Section 3 in which there was always a possibility of a local permutation of contexts corresponding to a local section. The Model-A is based on four elements in the measurement set represented by tetrahedral-description 12 whose symmetries is quantified in the presence of a topological space homeomorphic to a 4-torus 13. The Model-B is based on seven elements in measurement set representable as hexagonal pyramid 15 interacting with discrete surface homeomorphic to 10-torus.

4.1 Model-A

We are given with the tetrahedral-description of the model associated with discrete space of 4-torus in Figure 12 and Figure 13 respectively. The symmetries of the tetrahedron are subject to critical simplices (red) and virtual loops (blue) in its associated discrete space. One can construct the higher dimensional table from the given information of CS and VL set in a straightforward way as shown in Figure 14. Higher dimensional table can encode information about n-contexts simultaneously as in the table comprising of 3-contexts and 4-contexts along with classical 2-contexts. The table expresses a scenario in which there is possibility of no local and global sections unlike other models of Section 3 in which local sections are possible. Such scenarios could express stronger form of contextuality such that no variables of the system can be simultaneously measured, hence no possible local transformation between any vertices representing variables – hyper-contextual scenarios.

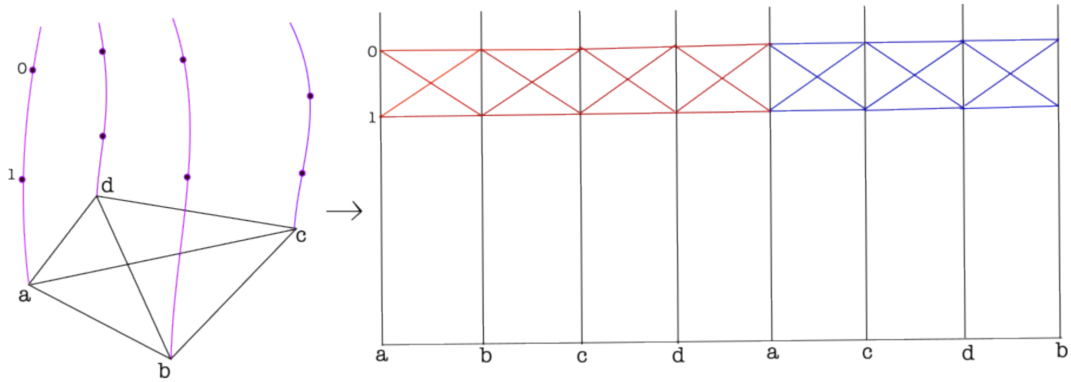


Figure 12: A given tetrahedral-description of Model-A.

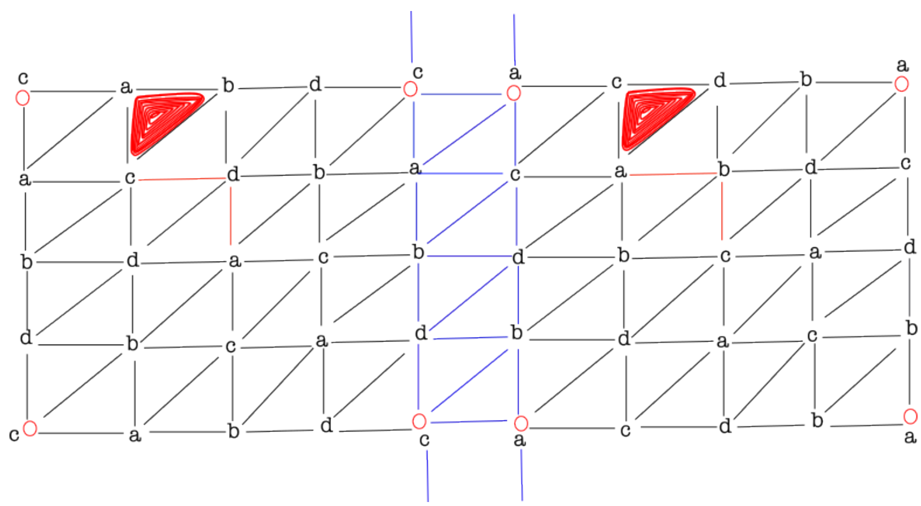


Figure 13: A given discrete space of genus 4 of Model-A.

	00	10	01	10		000	010	101	111		0000	1010	0101	1111
ab	0	1	1	0	abc	0	0	0	0	acdb	0	0	0	0
bc	0	1	1	0	acd	1	0	0	1					
cd	0	1	1	0	abd	1	0	0	1					
da	0	1	1	0	cbd	0	0	0	0					

Figure 14: Possibilistic table of Model-A computed from given hexagonal pyramid-description and its associated discrete space. Red colour corresponds to critical simplices of the space and blue colour corresponds possible virtual loops.

4.2 Model-B

We are given with the hexagonal pyramid-description of the model associated with discrete space homeomorphic to a 10-torus in Figure 15 and Figure 16 respectively along with its higher dimensional tabular representation in figure 17.

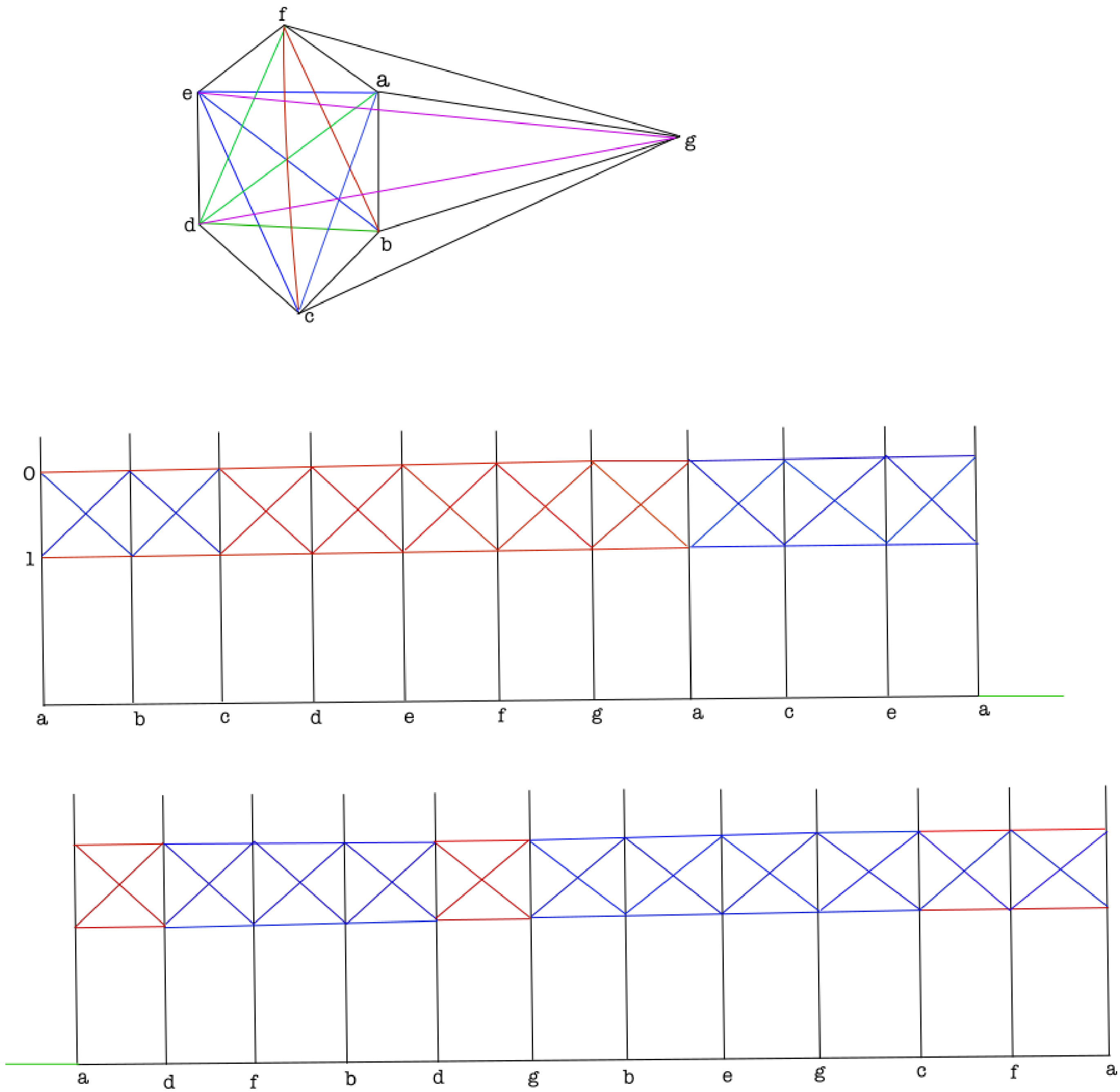


Figure 15: A given hexagonal pyramid-description of Model-B.

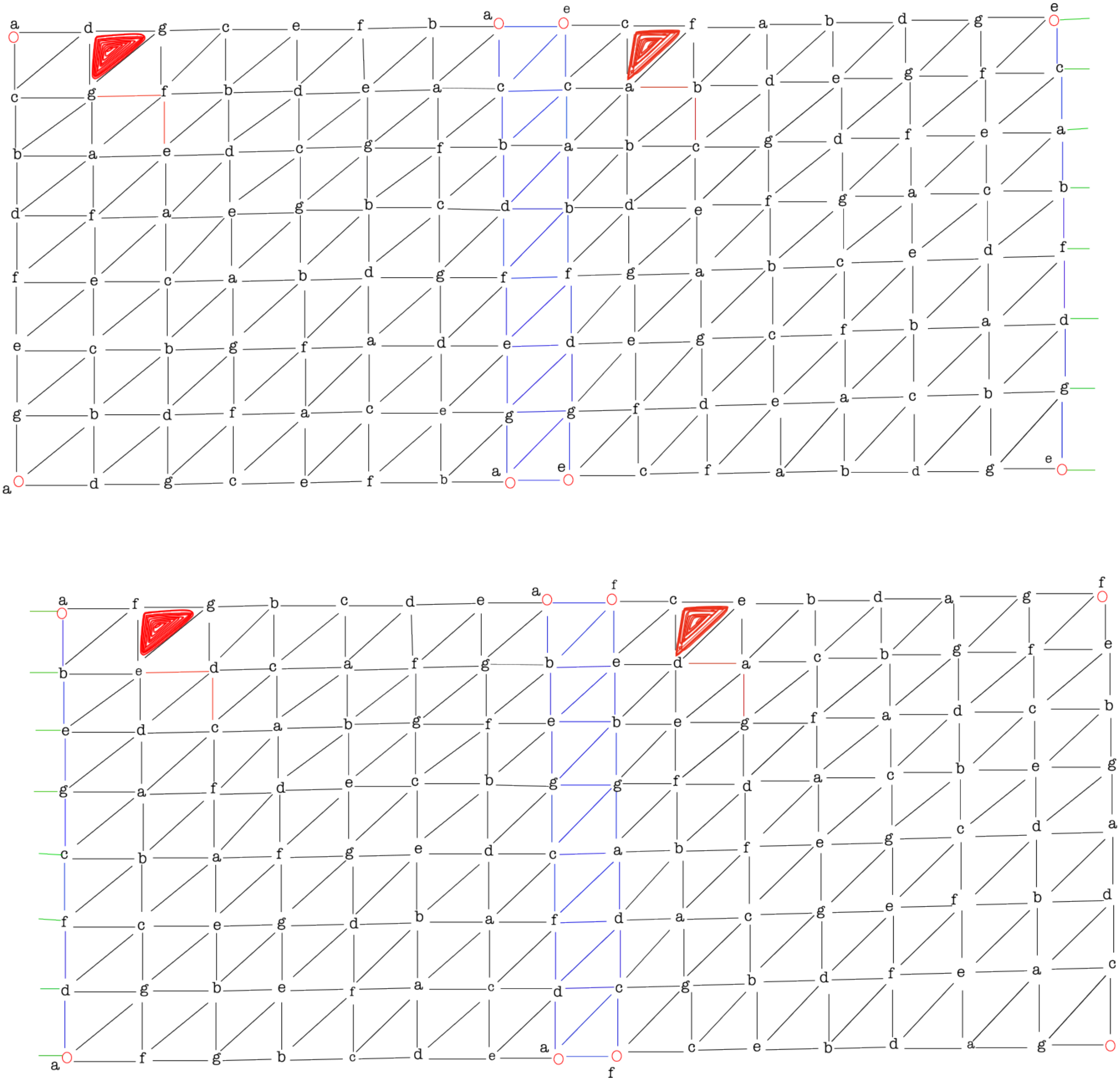


Figure 16: A given discrete space of genus 10 of Model-B.

	00	10	01	10		000	010	101	110	001	111		0000	1010	0101	1111
ab	0	1	1	0	abc	1	0	0	1	1	1	acea	0	0	0	0
bc	0	1	1	0	cde	1	0	0	1	1	1	dfbd	0	0	0	0
cd	0	1	1	0	efg	1	0	0	1	1	1	gbeg	0	0	0	0
de	0	1	1	0	cfa	0	0	0	1	1	0					
ef	0	1	1	0	cgc	1	1	1	0	0	1					
fg	0	1	1	0	gdg	1	1	1	0	0	1					
ga	0	0	0	0												
ad	0	0	0	0												

Figure 17: Possibilistic table of Model-B computed from given hexagonal pyramid-description and its associated discrete space. Red colour corresponds to critical simplices of the space and blue colour corresponds possible virtual loops.

5 Discussion

The formalism could facilitate quantifying novel algebraic- structures to quantify higher order contextuality. A starting point could be a seven dimensional Mermin-Ardehali-Belinskii-Klyshko model whose exclusivity graph is shown in Figure 18. The graph has 16 vertices that could be represented by square gyrobicupola polyhedral description to induce symmetry transformations between different contexts. Its Euler characteristics χ by formula including edges, vertices and faces turns to be 2 of a sphere. The higher simplices like tetrahedron shown in blue in Figure 18are not considered in evaluating χ that would stimulate a connection with rich geometric and topological structures over 3-manifold. The work of Atiyah and Bott on Morse theory and algebraic bundle theory [7], along with Thurston's work on geometry of 3-manifold [14] and Floer's homology [9] could be starting point for exploring these intriguing connections.

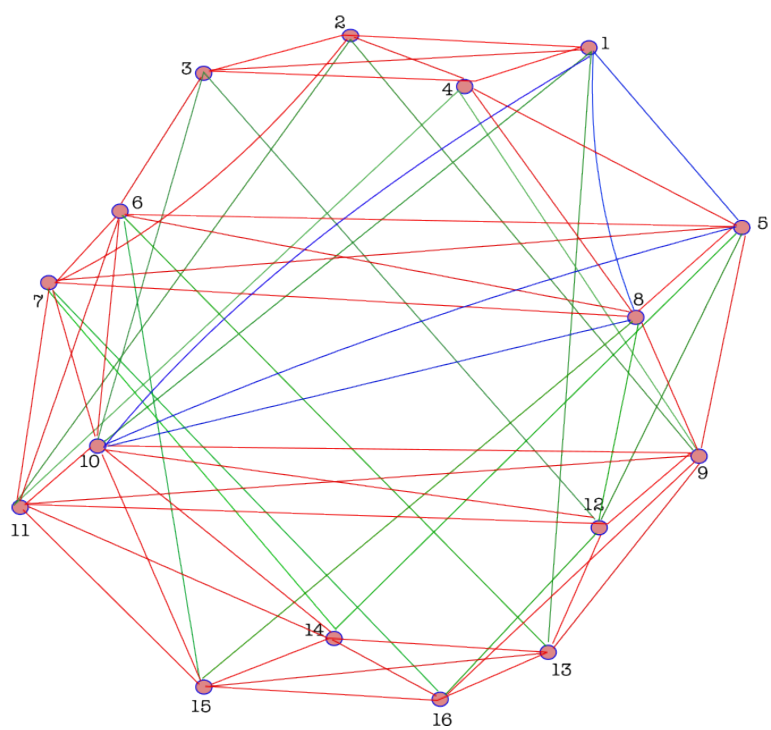


Figure 18: The exclusivity graph of seven dimensional Mermin-Ardehali-Belinskii-Klyshko model that could have square gyrobicupola polyhedral description and a spherical topology.

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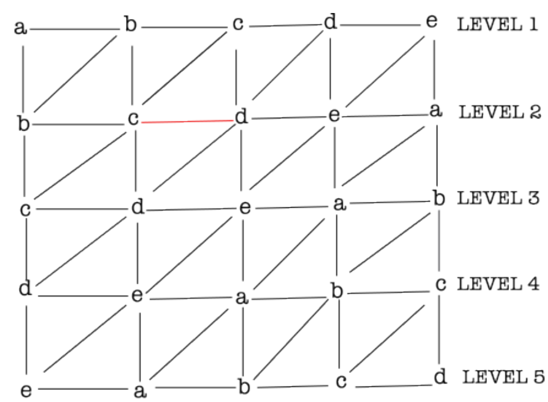
Appendices

A Additional Construction Information

Suppose we randomly chose context cd of level 2 of the combinatorial structure as shown in Figure 19. Following this set-up, ab_3 would mean ab of 3rd level. In order to compute the particular context cd of 2nd level (cd_2) among other eight possible $\mathrm{cd}'s$ at different level, one consider all loops and then map out which particular context of particular level is not considered in this loop-set, that would correspond to critical simplex. The interest in loops is due to fundamental group being the semantic description of the structure quantifying contextuality as non-collapsibility property of space.

Below are all loops while choosing a as starting vertex. Let L be set of all such paths $L = \sum_{n=1}^{34} L_n$. Any other path (if any) would be linear combination of paths traversed below. Moreover, these paths are sufficient for our case because it traversed all possible links between all vertices. Suppose $\mathbf{L}$ be a universal set of all paths of $\mathcal{K}_1$, then using complement or difference between $\mathbf{L}$ and L gives set of critical contexts. It turns out one context c_2d_2 is critical that was randomly chosen in $\mathcal{K}_1$. Any easy way to check is that in all the paths below there is no sequence with c_2d_2 which implies its criticality. On contrary discrete Morse function does not allow these loops due to its definition so we identify the boundary of $\mathcal{K}_1$ to topologise the disc into a surface of a torus $\mathcal{K}_2$ as in Figure 5.

$L_1 = a_1b_1c_1d_1e_1a_2b_3b_4c_4b_3a_2e_1d_1c_1b_1a_1$	$L_2 = a_1b_1c_2d_3c_3b_2a_1$	$L_3 = a_1b_1c_2d_3e_4d_4c_3b_2a_1$
$L_4 = a_1b_1c_2d_3e_4a_5e_5d_4c_3b_2a_1$	$L_5 = a_1b_1c_1d_2e_3d_3c_3b_2a_1$	$L_6 = a_1b_1c_1d_2e_3d_3c_2b_2a_1$
$L_7 = a_1b_1c_1d_2e_3d_3c_2b_1a_1$	$L_8 = a_1b_1c_1d_2e_3a_4e_4d_4c_3b_2a_1$	$L_9 = a_1b_1c_1d_2e_3a_4b_5a_5e_5d_4c_3b_2a_1$
$L_{10} = a_1b_1c_1d_2e_3d_3c_2b_1a_1$	$L_{11} = a_1b_1c_1d_2e_3d_3c_3b_2a_1$	$L_{12} = a_1b_1c_1d_2e_3a_4e_4d_3c_2b_2a_1$
$L_{13} = a_1b_1c_1d_2e_3a_4e_4d_3c_2b_1a_1$	$L_{14} = a_1b_1c_1d_2e_3a_4b_5a_5e_4d_3c_2b_2a_1$	$L_{15} = a_1b_1c_1d_2e_3a_4b_5a_5e_4d_3c_2b_1a_1$
$L_{16} = a_1b_1c_1d_1e_2a_3b_4a_4e_4d_4c_3b_2a_1$	$L_{17} = a_1b_1c_1d_1e_2a_3e_3d_3c_3b_2a_1$	$L_{18} = a_1b_1c_1d_1e_2a_3e_3d_3c_3b_1a_1$
$L_{19} = a_1b_1c_1d_1e_2a_3b_4a_4e_4d_4c_3b_2a_1$	$L_{20} = a_1b_1c_1d_1e_2a_3b_4a_4e_4d_3c_2b_1a_1$	$L_{21} = a_1b_1c_2b_2b_1a_1$
$L_{22} = a_1b_1c_1d_1d_2e_2e_3a_3a_4b_4b_5a_5e_5d_4c_3b_2a_1$	$L_{23} = a_1b_1c_2d_3e_4d_4d_3c_3c_2b_2b_1a_1$	$L_{24} = a_1b\|_1c_1d_2d_3c_2b_2a_1$
$L_{25} = a_1b_1c_1d_1e_2a_3b_4c_5b_5a_5e_5d_4c_3b_2a_1$	$L_{26} = a_1b_1c_1d_1e_2d_2c_1b_1a_1$	$L_{27} = a_1b_1c_1c_2b_1a_1$
$L_{28} = a_1b_1c_2d_3e_4a_5e_5e_4d_4d_3c_3c_2b_2b_1a_1$	$L_{29} = a_1b_1c_1d_2e_3e_4a_4a_5e_5d_4c_3b_2a_1$	$L_{30} = a_1b_1c_2d_3c_3c_2b_1a_1$
$L_{31} = a_1b_1c_1d_1e_1a_2e_2e_1d_1c_1b_1a_1$	$L_{32} = a_1b_1c_1d_1e_1a_2b_3a_3a_2e_2e_3d_3c_3b_2a_1$	$L_{33} = a_1b_1c_2b_2a_1$
$L_{34} = a_1b_1c_1d_1e_1b_3c_4c_5b_4a_3e_2d_1c_1b_1a_1$		



$\mathcal{K}_1$

Figure 19: The combinatorial description of the physical set up of the experiment with a randomly chosen critical context cd

B Discrete Morse Theory:

Discrete Morse theory (DMT) over a finite simplicial complex $\mathcal{K}$, with $\mathcal{S}_p$ the simplices of dimension p. Discrete Morse function $\mathfrak{f}$ assigns a single real number to each simplex of $\mathcal{K}$. A function $\mathfrak{f} : \mathcal{K} \to \mathbb{R}$ be a discrete Morse function iff, for every d dimensional simplex $\sigma^d \in \mathcal{K}$, the following two conditions hold: $|\{\tau^{(d+1)} > \sigma^{(d)} \mid \mathfrak{f}(\tau^{(d+1)}) \leq \mathfrak{f}(\sigma^{(d)})\}| \leq 1$ and $|\{\tau^{(d-1)} < \sigma^{(d)} \mid \mathfrak{f}(\tau^{(d-1)}) \geq \mathfrak{f}(\sigma^{(d)})\}| \leq 1$. σ is called a critical simplex if such conditions don't hold. The function decreases from 2-simplex to one simplex and vice-versa in reverse direction. Forman's beautiful papers [10] are the standard references on this subject. There is a relation between critical simplices and collapsibility of simplicial complex. If there is no critical simplices in an interval say $[a, b]$ then it is collapsible to a vertex and the space is simply connected. Moreover, the number of critical points of some index are responsible for the topology of the underlying structure. However, one is unable to determine the efficiency of this discrete Morse function with respect to collapsibility i.e; there may be simplices that are critical that could be collapsed while still preserving the homotopy type. A simple example is that of a sphere. So, it is not only critical simplices that are significant in determining the collapsibility of topological space but also the paths between simplices. For example, if there is only one gradient path between two simplices then reversing the direction of the path cancels criticality of both the simplices. The criticality in $\mathcal{K}$ induces non-triviality in paths up to homotopy. Direction (arrows) over simplicial complex can be defined by introducing the concept of combinatorial vector field. It gives direction to each of the simplicial complex. A combinatorial vector field is a map $V : \mathcal{S} \to \mathcal{S} \cup \{0\}$. Given such a map V and $\sigma \in S$ with $V(\sigma) \neq 0$ we draw an arrow on $\mathcal{K}$ whose tail begins at σ and which extends into $V(\sigma)$. It satisfies some properties like if V implies that σ is always face of $V(\sigma)$ then an arrow is possible. If τ is the head of an arrow then it cannot be a tail as well. Moreover every simplex is head and tail of, at most, one arrow. This field classifies each simplex in pairs and there are three disjoint possibilities viz., σ is the head of an arrow ($\sigma \in Image(V)$), σ is the tail of an arrow ($V(\sigma) \neq 0$) and σ is neither head nor tail of any arrow ($V(\sigma) \neq 0$) and $\sigma \notin Image(V)$ (critical simplices).The non-critical simplices occur in pairs $\sigma^{(d)} \subset \tau^{(d+1)}$ where $\mathsf{f}(\sigma^{(d)}) \geq \mathsf{f}(\tau^{(d+1)})$. It can be illustrated by drawing an *arrow* from $\sigma^{(d)}$ to $\tau^{(d+1)}$. The points that are neither heads nor tails of any arrow are exactly the critical simplices. A combinatorial vector field V on $\mathcal{K}$ is a collection of pairs $\sigma^{(d)} \subset \tau^{(d+1)}$ of simplices of $\mathcal{K}$ such that each simplex belongs to at most one pair of V. If α is not critical then there is a unique edge $e > \alpha$ with $f(e) \leq f(\alpha)$ i.e., $V(\alpha) = e$ with chosen orientation. If α is critical $V(\alpha) = 0$. Given a vector field V on a simplicial complex K, a V-path is a sequence of simplices $\sigma_0^{(d)}, \tau_0^{(d+1)}, \sigma_1^{(d)}, \tau_1^{(d+1)}, \sigma_2^{(d)}, ..., \tau_r^{(d+1)}, \sigma_{r+1}^{(d)}$ such that for each $i = 0, ..., r$, $\{\sigma, \tau\} \in V$ and $\tau_i > \sigma_{i+1} \neq \sigma_i$. We say such a path is a *non-trivial closed path* if $r \geq 0$ and $\sigma_0 = \sigma_{r+1}$. If V is a vector field on a simplicial complex K, V is the gradient vector field of some discrete function on $\mathcal{K}$ if and only if there are no non-trivial closed V-paths i.e., Hasse diagram is acyclic and $\mathcal{K}$ is *collapsible.* If V has (non-trivial) closed paths, then it cannot be the gradient of a function. It corresponds to the transformation of the fundamental group of topological space.

C Construction proofs of Empirical Models

The section applies the method of Section 3 to other different examples of empirical models of quantum physics. Apart from the Liar cycle model, we shall give examples of tetrahedron model, Bell model, Hardy model, trivial model, Klyachko-Can-Binicioglu-Shumovsky (KCBS) model and Svetlichny's box.

C.1 Trivial model

We start with a basic example of a trivial model with no non-possibilistic events as shown in Figure 20.

	00	01	10	11
ab	1	1	1	1
bc	1	1	1	1
cd	1	1	1	1
da	1	1	1	1

Figure 20: Possibilistic table of trivial model.

All the events of the model are feasible so there is no requirement to include any other class of behaviour. As a result, the discrete space associated with a trivial model has a real line or disc topology describing all possible combinations of set-up. $\mathcal{M} = \{a, b, c, d\}$ has a discrete topology shown in Figure 21 and $|\mathcal{M}| = 4$ which equips it with tetrahedral description as in Figure 22.

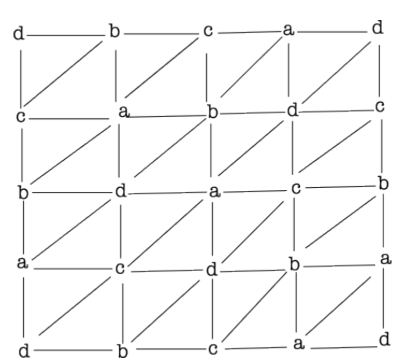


Figure 21: Discrete space of trivial model is homeomorphic to a disc. The table has all events as possibilistic which means it does not put any constraint on possible symmetries of tetrahedron representing its contexts.

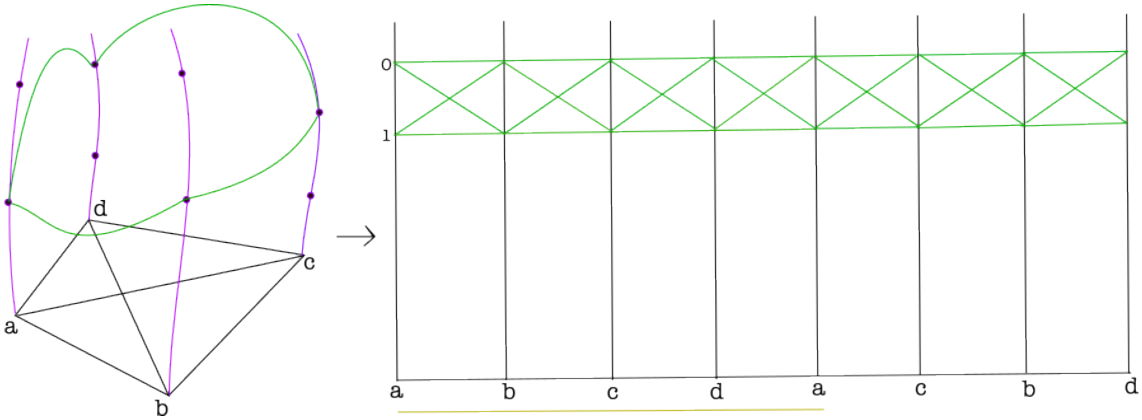


Figure 22: The tetrahedral description of trivial model on the left hand side and its linearised planar diagram on left hand side. The underlying space has no set of critical simplices and set of virtual loops as in Figure 21, so there exists a family of global sections one of which is represented in yellow starting and ending at vertex a at 11 on right hand side of figure.

Remark. *A model with a set of non-possibilistic events represented as* 0*'s can be considered trivial, i.e., model in which all sections are feasible, provided one does not include these events in the model description as discussed in Figure 23.*

(i)

	00	01	10	11
ab	1	0	0	1
bc	1	0	0	1
cd	1	0	0	1
de	1	0	0	1
ea	1	0	0	1

(ii)

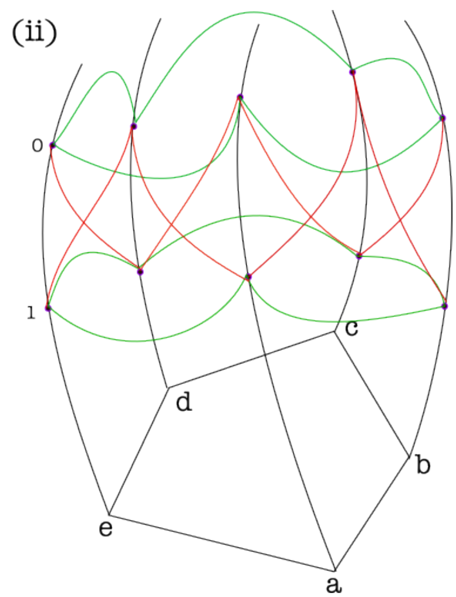


Figure 23: A model that has all possibilistic events represented as 1's is a conventional trivial model. The triviality also depends on the class of considered events from the table along with their corresponding paths. For instance, above table on (i) has many non-possibilistic events represented as 0, but the considered set of events are contexts at 11 and 00 and those at 01 and 10 are not considered. As a result, its bundle diagram has possible global sections starting and ending at a represented in green colour. The red lines are paths that are not part of the considered events, hence turns a trivial model by specifying only possibilistic component of events.

C.2 Bell Model

A simple logically contextual model is Bell model, i.e., there are possible compatible family of global sections whose topological space is shown in Figure 25 and tetrahedral description representing sections is shown in Figure 26. There are two non-possibilistic events in the Bell model as shown in table of Figure 24 unlike trivial model where every event was possibilistic.

	00	01	10	11
ab	1	0	0	1
bc	1	1	1	1
cd	1	1	1	1
da	1	1	1	1

Figure 24: Possibilistic table of the Bell model.

These two non-possibilistic events are expressed over 1-torus, so there is no need to iterate the space, hence no emergence of virtual loops at its boundary.

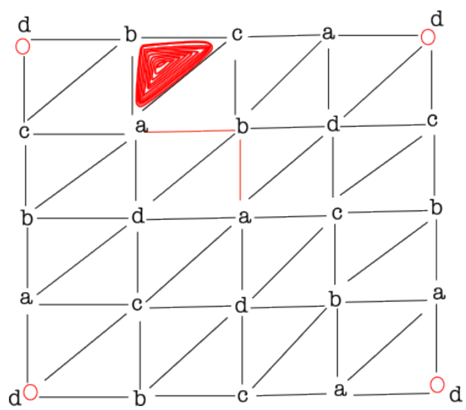


Figure 25: Topological space associated with the Bell model homeomorphic to 1-torus.

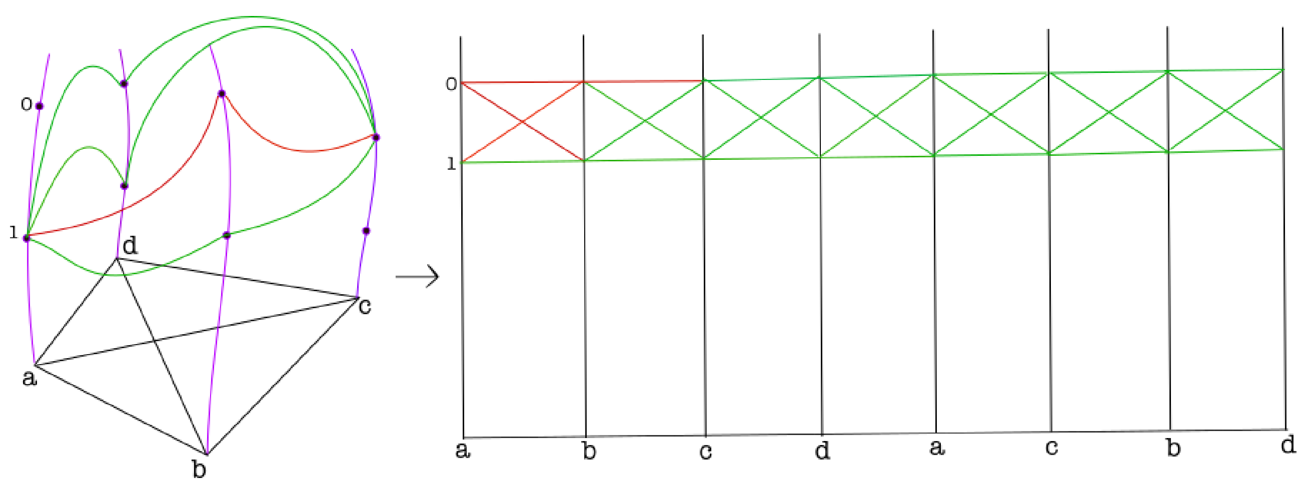


Figure 26: The tetrahedral description of model ($|\mathcal{M}| = 4$). There exist family of global sections in the Bell model, hence logically contextual. The critical simplices set is *ab*, *ab* and *abc* as shown in Figure 25 constrains the permutation of these contexts, hence shown as red in linearised planar diagram. Example of a path leading to global section and another path not leading to global section is given over fiber space of tetrahedron.

C.3 Hardy Model

The possibilistic table of the Hardy model is shown in Figure 27 and its polyhedral description is shown is Figure 28, whose symmetries are constrained by its associated discrete space is constructed in Figure 29.

	00	01	10	11
ab	1	1	1	0
bc	0	1	1	0
cd	1	1	1	0
de	0	1	1	0
ea	1	1	1	0

Figure 27: Possibilistic table of the Hardy model.

Remark. *The position of the* 0*'s and* 1*'s in the table is a characteristic feature to describe the strengths of contextuality, i.e., changing the position of* 0 *or* 1 *in the table can lead to different results. For instance, changing a non-possibilistic event at context de at* 11 *to* 10 *of the Hardy model could inhibits one of family of its global sections starting and ending at d represented in pink colour in Figure 28 which is otherwise a global section of the Hardy model.*

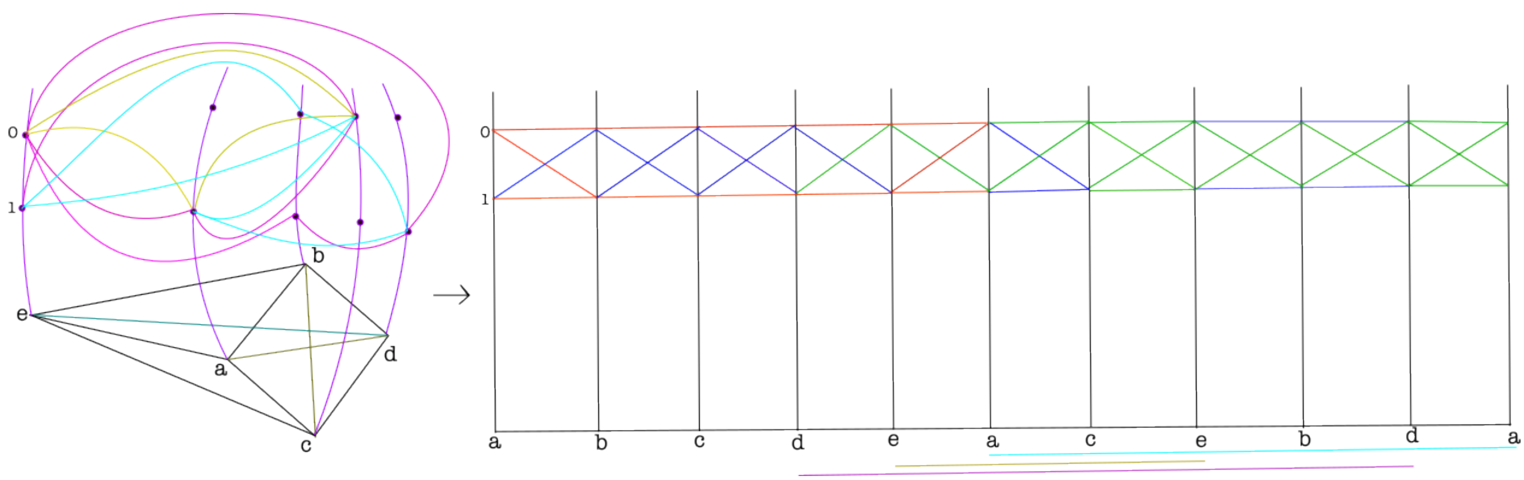


Figure 28: The symmetry transformations constrained via set of critical simplices and virtual loops set of Figure 29. The CS-set is { bcc, cd, dc, cdd, de, ea, eaa, ab, bc, abb, bc, cd} and VL-set is { abb, bbc, cbc, ccd, dcd, dde, dee, eea, aea, aab, bdc, dce, ced, edb, bde, bea, aea, aac, acb, cbd, dee, eea, eab, abb, bba, bac, acc, ccd, cdd, dde}. The model is logically contextual, i..e, there are three class of possible global sections represented by pink, cyan and yellow colour over fiber space of the square pyramid on left hand side of the figure. These three global paths are easy to see in its linearised planar diagram in right side of the figure denoted by pink line, cyan line and yellow line corresponding to global sections starting and ending from d, a and e respectively.

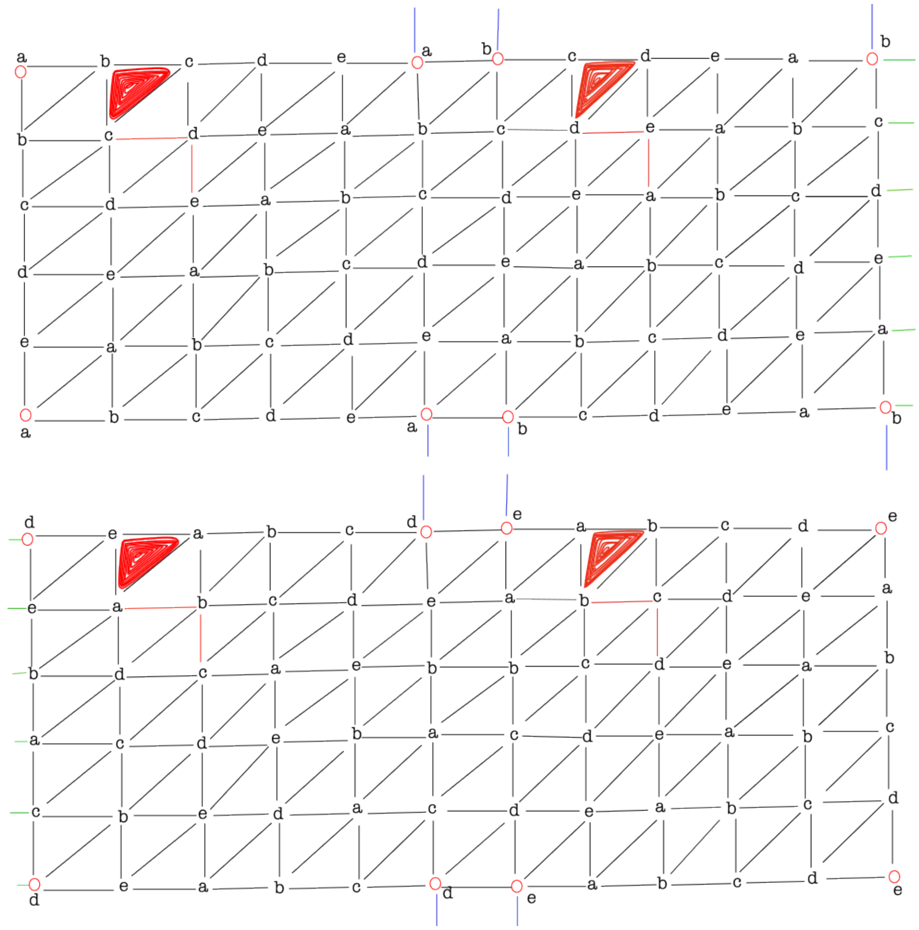


Figure 29: Discrete surface of genus = 4 is maximal surface to express class of behaviour, i.e., possibilistic, non-possibilistic and LC-GI via generic simplices (black), critical simplices (red) and virtual loops (blue). The space constrains the symmetry transformations of its polyhedral description as in Figure 28.

C.4 Tetrahedron Model

The possibilistic table of the tetrahedron model is shown in Figure 33 and its polyhedral description is shown is Figure 32, whose symmetries are constrained by its associated discrete space is constructed in Figure 31.

	000	001	010	011	100	101	110	111
abc	1	0	0	1	0	1	1	0
abd	1	0	0	1	0	1	1	0
acd	0	1	1	0	1	0	0	1
bcd	0	1	1	0	1	0	0	1

Figure 30: Possibilistic table of tetrahedron model.

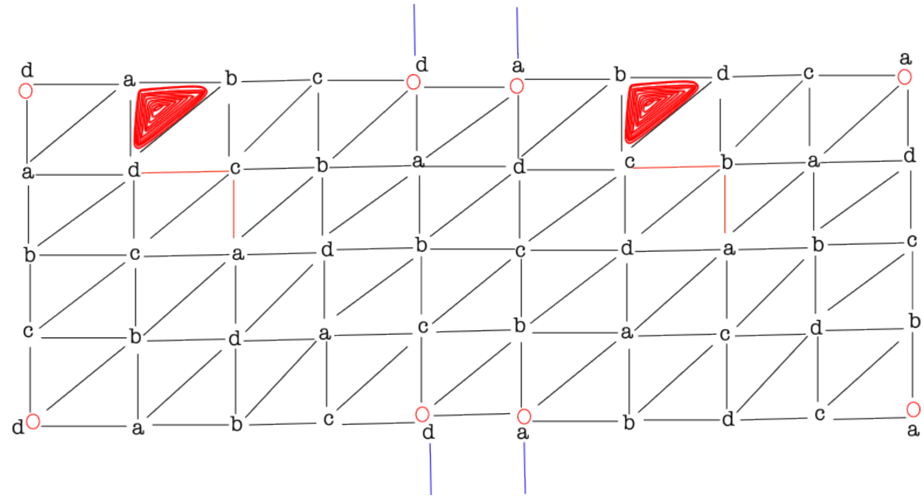


Figure 31: The discrete space homeomorphic to a surface of genus = 8 is a maximal space associated with the tetrahedron model. The space is actually 4 times in the same fashion to consider all possible behaviour of the table but are not completely shown for brevity.

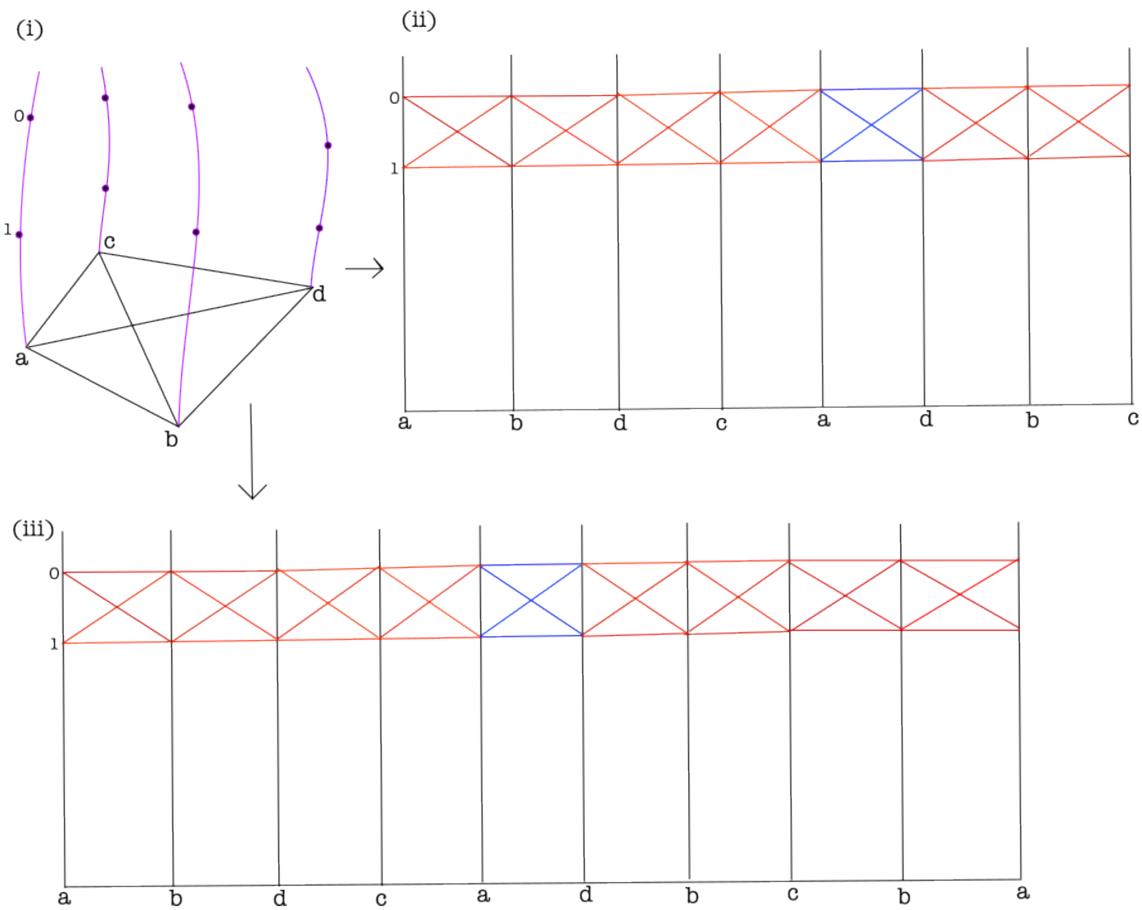


Figure 32: A tetrahedral description of the model as shown in (i) of the figure. (ii) is a linearised version of the tetrahedron consists of all edges to quantify its all possible symmetries: no possible global family of sections exist, hence strongly contextual model. (ii) is a sufficient quantification of symmetry between contexts and takes into account all possible of permutations between contexts, but a context *cba* is not explicitly described due to the process of linearising the tetrahedron. One can add this context *cba* in right order to include this context for visualisation as shown in (iii) of the figure. The red and blue coloured transitions are based on CS-set and VL-set respectively. CS-set = { abd, abd, abd, abd, dca, dca, dca, dca, abc, abc, abc, abc, dbc, dbc, dbc, dbc } and VL-set = { daa, aad, adb, dbc, cbc, ccb, cbd, bda, daa, aad, adb, dbc, cbc, ccb, cbd, bda, add, dda, adc, acb, cbb, bbc, cba, cad, add, dda, adc, acb, cbb, bbc, cba, cab }

C.5 Klyachko-Can-Binicioglu-Shumovsky Model (KCBS)

	00	01	10	11
ab	0	1	1	0
bc	0	1	1	0
cd	0	1	1	0
de	0	1	1	0
ea	0	1	1	0

Figure 33: Possibilistic table of tetrahedron model.

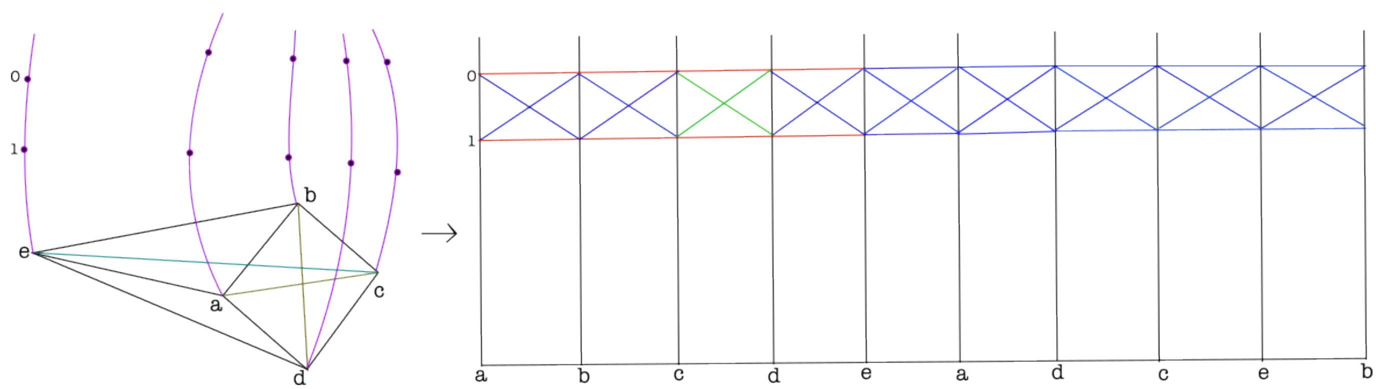


Figure 34: A polyhedral description (square pyramid) of the model as shown in (i) along with its linearised version in (ii), no possible global family of sections exist, hence strongly contextual model. The CS-set is {bcc, cd, de, cdd, de, ea, dee, ea, ab, eaa, ab, bc, abb, bc, cd} and possible VL-set { abb, bbc, cbc, ccd, dcd, dde, ede, eaa, aab, cdd, dde, dee, eea, eaa, aab, abb, bbc, bcc, ccd, bcc, ccd, cdd, dde, dee, eea, eaa, aab, abb, bbc, dee, eea, eaa, aab, abb, bbc, cbc, ccd, ddc, dde} Apart from the relevant critical simplices corresponding to non-possibilistic events of the table, the relevant combinations of the virtual loop set in terms of polyhedral symmetry are: { abc, abc, deaed, eae, adad, dcdc, cece, adcc ebbe}. These are possible linear combinations of loops in its discrete space 35. We have not considered all linear combinations of possible virtual loops for brevity but provided core elements in the set.

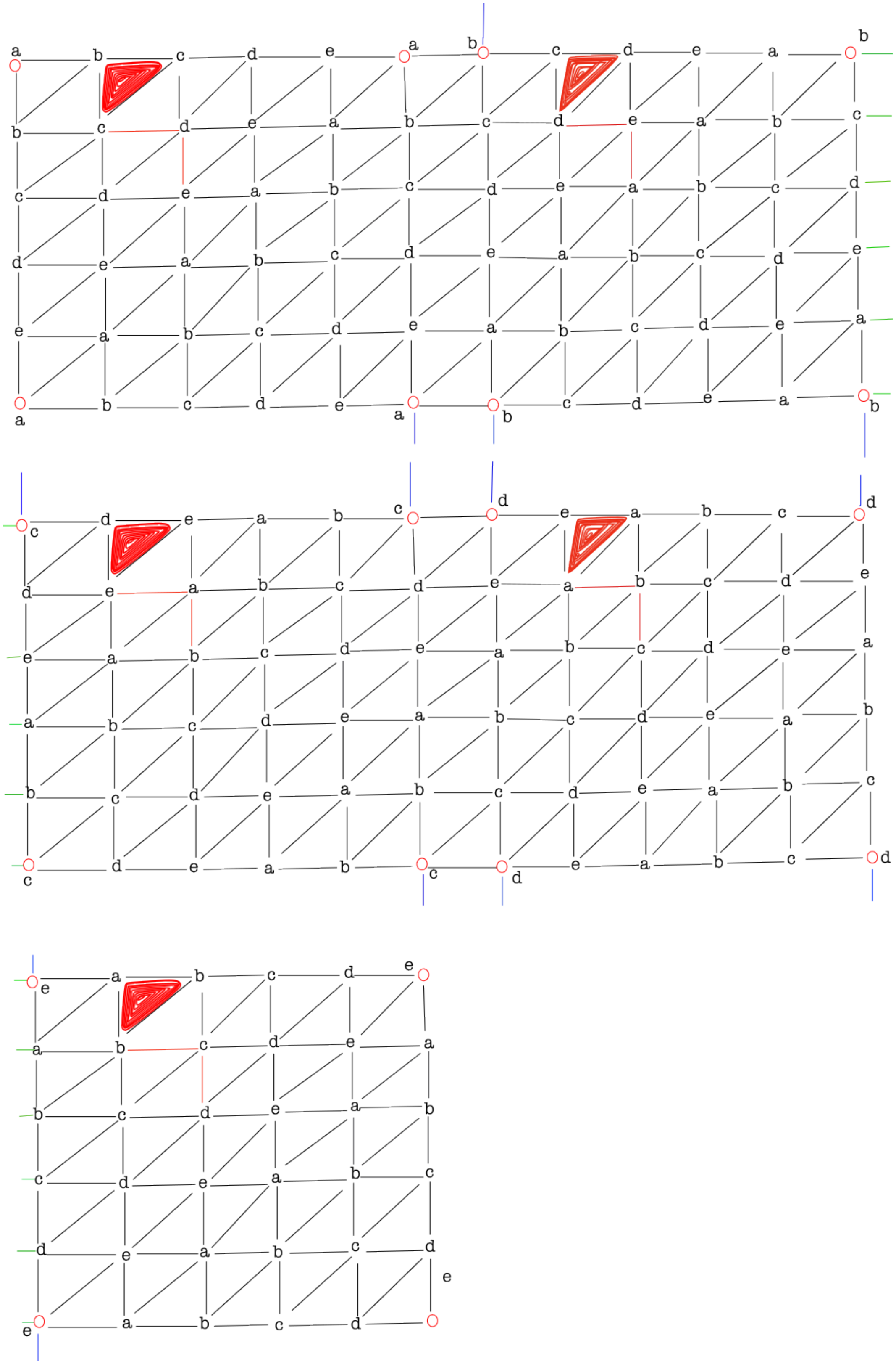


Figure 35: Discrete surface of genus = 5 is maximal surface to express class of behaviour, i.e., possibilistic, non-possibilistic and LC-GI via generic simplices (black), critical simplices (red) and virtual loops (blue). The space constrains the symmetry transformations of its polyhedral description as in Figure 34.

C.6 Svetlichny's Box

	000	001	010	011	100	101	110	111
ABC	1	0	0	1	0	1	1	0
ABc	1	0	0	1	0	1	1	0
AbC	1	0	0	1	0	1	1	0
Abc	0	1	1	0	1	0	0	1
aBC	1	0	0	1	0	1	1	0
aBc	0	1	1	0	1	0	0	1
abC	0	1	1	0	1	0	0	1
abc	0	1	1	0	1	0	0	1

Figure 36: Possibilistic table of Svetlichny model.

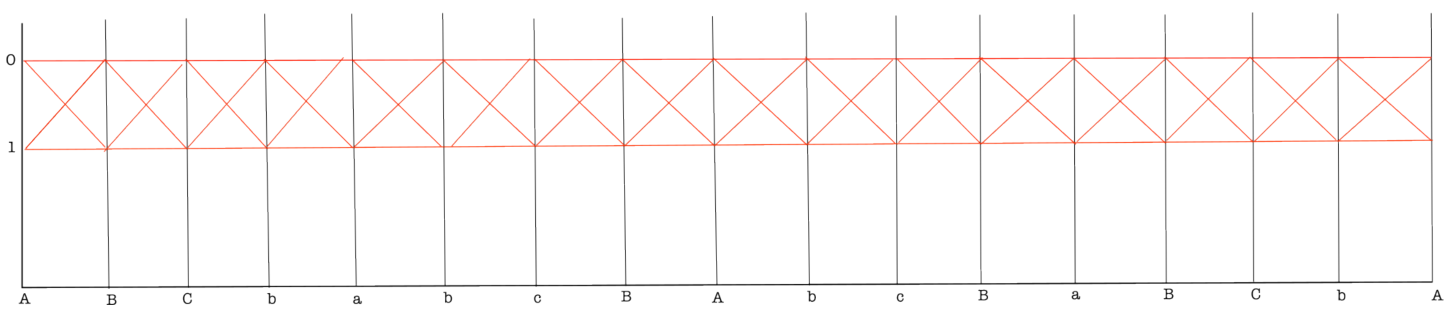


Figure 37: Polyhedral description of the model, no global section exists, hence strongly contextual in nature. We have provided only relevant contexts here. All possibilities of permutations are considered in Appendix **??**. The *CS*-set is { ABC, Abc, ABc, abc, aBC, CbA, bCa, aBc }. Each element of *CS*- set iterates 4 times. The *VL*-set is { ACB, CBa, BaC, Cca, cab, abb, bbB, bBc, BAc, AcA, AAC, CAa, AaB, aBc, BcC, Ccb, Cba, baB, aBb, BbA, Abc, AcC, cCA, AaB, Bba, BbC, bCA, ACa, AaB, Bab, BbC, bCc, Ccc, ccA, AAc, aAb, AbB, BbA, ABC, ABC, BCa, BaC, aCb, Cbc, bcc, cca, caA}

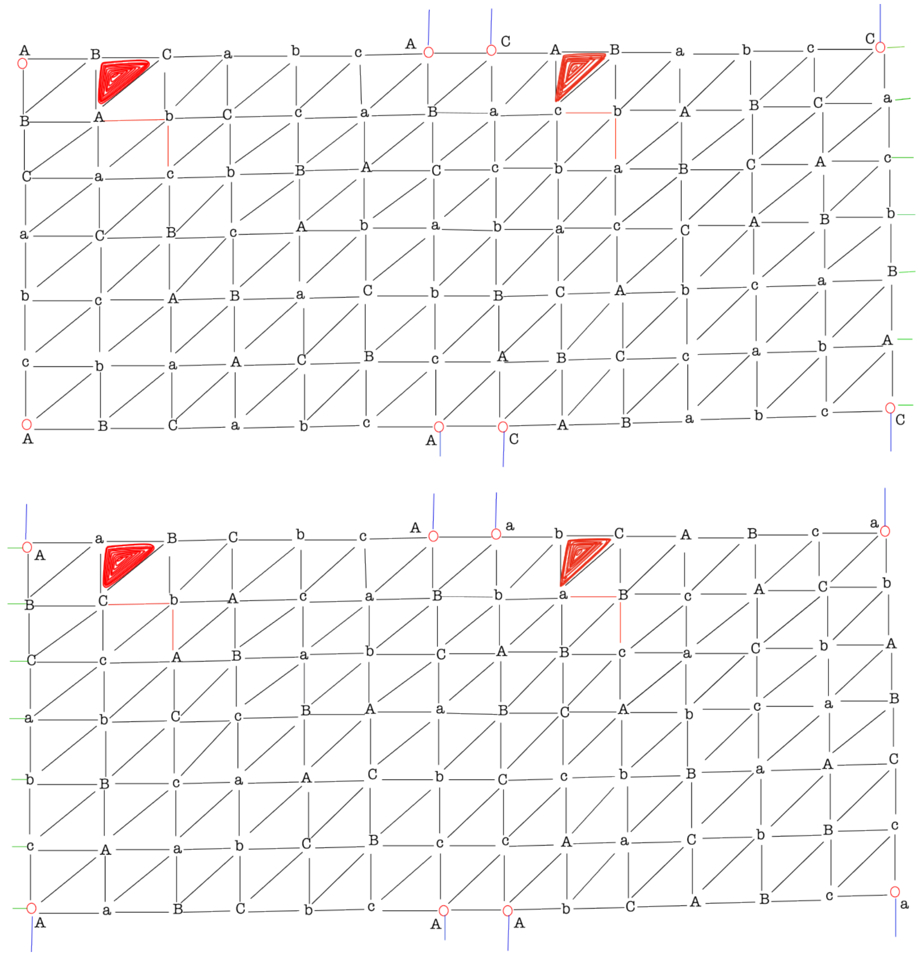


Figure 38: Discrete surface of genus = 8 is maximal surface to express class of behaviour constraining its polyhedral description as shown in Figure 37. The exact space is iterated once more which is not shown for brevity.

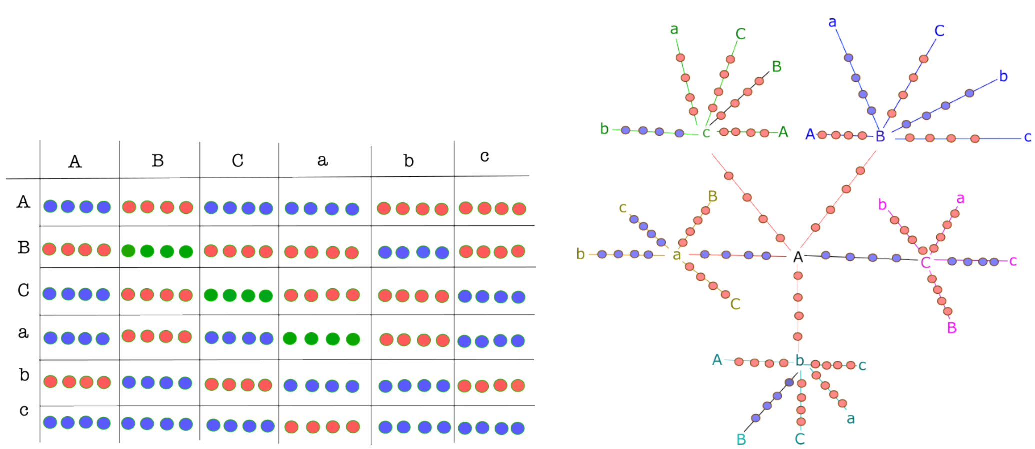


Figure 39: The figure represents all possible permutations of the Svetlichny model. There are six vertices of the pentagonal pyramid each of which can be permuted with other five vertices and to itself as well as each permutation can have four possibilities in the fiber space represented with four (coloured) circles over each permutation as shown in the left side of the figure. The red and blue coloured circles represent critical simplices and virtual loops which are given from its discrete space of Figure 38 and green coloured circle is the possibilistic/feasible permutation. On the right side of the figure is a clear view of all possibilities which looks like a big flower consisting of seven petals corresponding to permutations of six vertices. Note that a local section is a transition between any green coloured circles of any petals and the global section is the loop that should start from a green coloured circle of a specific petal and reach back to it. It is clear that there are local sections but there is no family of global sections.